2.4 Circuits with Resistors and Capacitors

- response of a series connected resistor and capacitor to a dc (steady) voltage
- response of a series connected resistor and capacitor to an ac (varying) voltage
- decibels and Bode plots
- high-pass electronic filter
- band-pass electronic filter
- using a low-pass filter for signal-to-noise enhancement
Response of an RC Circuit to DC

Consider the circuit at the right. When the switch is connected to the battery, current will flow and charge the capacitor. The charging will continue until the voltage across the capacitor equals the voltage of the battery.

For $V = 9 \, \text{V}$, $R = 10 \, \text{k}\Omega$ and $C = 1 \, \text{mF}$, the current can be computed as a function of time by tracking the total capacitor charge.

<table>
<thead>
<tr>
<th>time (s)</th>
<th>charge (mC)</th>
<th>$V_C$ (V)</th>
<th>$V_R$ (V)</th>
<th>$i$ (mA)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0 mC/1 mF = 0</td>
<td>9</td>
<td>0.9 mA</td>
</tr>
<tr>
<td>1</td>
<td>0+0.9</td>
<td>0.9 mC/1 mF = 0.9</td>
<td>8.1</td>
<td>0.81</td>
</tr>
<tr>
<td>2</td>
<td>0.9+0.81</td>
<td>1.71 mC/1 mF = 1.71</td>
<td>7.29</td>
<td>0.73</td>
</tr>
<tr>
<td>4</td>
<td>1.71+0.73</td>
<td>2.43 mC/1 mF = 2.43</td>
<td>6.57</td>
<td>0.66</td>
</tr>
<tr>
<td>$\infty$</td>
<td>9 mC</td>
<td>9</td>
<td>0</td>
<td>0 mA</td>
</tr>
</tbody>
</table>
The figure at the right shows the voltage across the resistor (blue) and capacitor (red) when a square voltage pulse is applied to the circuit on the previous slide. The time dependence of the current is derived as follows.

\[
V = V_C + V_R \\
V = \frac{Q}{C} + iR \\
\frac{dV}{dt} = \frac{1}{C} \frac{dQ}{dt} + R \frac{di}{dt} = 0 \\
\int_{0}^{t} \frac{di}{i} = -\frac{1}{RC} \int_{0}^{t} dt
\]

The time dependence of the two voltages is given by the following.

\[
Ri = Ri_0 e^{-t/RC} \\
V_R = V e^{-t/RC} \\
V_C = V - V_R =
\]
What is the impedance of an RC voltage divider when measuring across the capacitor? Start with resistance and capacitive reactance,

\[
\frac{X_C}{R + X_C} = \frac{-j/2\pi f C}{R - j/2\pi f C}
\]

solve for the impedance,

\[
|Z_{RC}| = \sqrt{\frac{-j/2\pi f C}{R - j/2\pi f C}} \cdot \frac{j/2\pi f C}{R + j/2\pi f C} = \frac{1}{\sqrt{1 + (2\pi RC f)^2}}
\]

and finally, define a characteristic frequency, ________________.

\[
|Z_{RC}| = \frac{1}{\sqrt{1 + (f/f_{3dB})^2}}
\]

The impedance is frequency dependent with the equation constant depending upon RC.
Voltage Across the Capacitor

\[ G \equiv \frac{V_C}{V} = |Z_{RC}| \]

as \( f \to 0, \ G \to 1 \)

as \( f \to \infty, \ G \to 0 \)

when \( f = f_{3dB} \)

When voltage is measured across the capacitor, the series combination of a resistor and capacitor is called a low-pass filter. Note that the gain of the circuit does not decrease rapidly. At frequencies larger than \( f_{3dB} \) the functional form approaches ______.
Phase Shift Between V and $V_C$

- put the voltage divider expression into the standard form, $a + j b$

$$\frac{X_C}{R + X_C} = \frac{-j/2\pi f C}{R - j/2\pi f C} = \frac{1}{1 + (2\pi R C f)^2} - j \frac{2\pi R C f}{1 + (2\pi R C f)^2}$$

- derive an expression for the phase angle using the arctangent in the complex plane

$$\phi = \tan^{-1} \left( \frac{\text{Im}}{\text{Re}} \right) = \tan^{-1} \left( \frac{-2\pi R C f}{1 + (2\pi R C f)^2} \right) = \tan^{-1} \left( -2\pi R C f \right) =$$

when $f = 0$, $\phi = 0^\circ$; when $f = f_{3\text{dB}}$, $\phi = -45^\circ$; as $f \to \infty$, $\phi \to -90^\circ$

- the phase shift is close to $0^\circ$ at frequencies up to $f_{3\text{dB}}$
- the phase shift is close to $-90^\circ$ at frequencies above $f_{3\text{dB}}$
- the phase shift varies significantly from $0.1f_{3\text{dB}}$ to $10f_{3\text{dB}}$
- phase shifts distort the shape of the signal
Decibels and the Low-Pass Bode Plot

- graphs of amplitude or power gain versus frequency are usually in decibels
  \[ A(dB) = 10 \log \left( \frac{P_{out}}{P_{in}} \right) = 10 \log \left( \frac{V_{out}^2}{V_{in}^2/R} \right) \]

- A is called attenuation for \( G < 1 \) and amplification for \( G > 1 \)
- for a large range of frequencies a Bode plot is easier to interpret than a linear plot
- one plot represents all low-pass filters
- A is flat below \( f_{3dB} \) and drops by 20 dB per decade above \( f_{3dB} \)
for a high-pass filter voltage is measured across the resistor

\[ |Z| = \sqrt{\left( \frac{R}{R + X_C} \right) \left( \frac{R}{R + X_C} \right)^*} \]

as \( f \to 0, G \to 0 \); as \( f \to \infty, G \to 1 \); when \( f = f_{3dB} \), \( G = 1/\sqrt{2} \)
A band-pass filter can be created by feeding the output of a low-pass filter into a high-pass filter.

\[ G_{BP} = G_{LP} G_{HP} \]

\[ A_{BP}(dB) = \square \]

The values of R and C do not need to be the same for the two parts of the filter. This permits the creation of a "flat" gain between the low-pass and high-pass 3dB frequencies (not shown).
Signal-to-Noise Enhancement

As the RC time constant is increased, the _____ decreases and ________ increases. 

This is an ever-present trade in instrument design.