1. Consider the odd temporal waveform pulse train given below. It extends from \(-\infty\) to \(+\infty\) with a repetition period of 400 seconds.

The goal of the question is to develop the spectrum using appropriate members of the Fourier transform basis set. The approach will use two steps: (1) develop the transform for the center pulse (the odd rectangle between -200 and +200 seconds) by ignoring the other repetitions; and, (2) take into account the repetitions.

(a 20 pts) Use four different ways to combine basis set transform pairs to obtain the spectrum of the center odd rectangle. Assume that for \(t < -100\) and \(t > +100\) seconds the amplitude is zero. There may be more than four ways. Of the four ways I developed, one of them had an intractable convolution integral. Leave that answer in a form having the correct unsolvable integral.

(b 5 pts) Demonstrate the correctness of your answer by creating a digital representation of the data waveform and determining the spectrum using the \(\text{odd}(f, \text{signal})\) function you created as part of Exam 2, Question 4. The temporal waveform should have values every 1 second, and have amplitudes restricted to the set \{-1, 0, +1\}. Let the waveform run from -500 to +500 seconds. Make a graph that shows your spectrum derived in part (a) along with the output of \(\text{odd}(f, \text{signal})\).

(c 10pts) Now take into account the waveform repetition. Look at slides 7.9-5,6 as a guide. Use the figure above to obtain the repetition period. Convert the period to a repetition frequency, then determine the comb frequencies. Make a vector that contains only those signal frequencies existing over the range -0.125 to +0.125 Hz. In part (a) you have determined the functional form of the middle waveform on slide 7.9-6. Use this function to compute the amplitude for each of the signal frequencies. Put them in a second vector. Convert this frequency and amplitude information into a Fourier Series by using the summation operator.
Remember that since the waveform is odd, the series is composed of sines and not cosines. Graph your Fourier Series to see how well it approximates the waveform given at the start of this problem.

2. This question explores the origins and extent of distortion caused by low pass RC filters.

(a 10pts) Create a temporal function that generates a Gaussian pulse train, where each pulse has a characteristic time, \( t_0 = 10 \text{ seconds} \), and a pulse spacing of 100 seconds. This is accomplished by using a cosine Fourier series. For this waveform 31 harmonics from \( n = -15 \) to \(+15\) (including zero), will suffice to give an excellent approximation of the Gaussian pulses. Make sure you normalize the function correctly so that the amplitude at the center of each peak is 1.00. Plot your function from -250 to +250 seconds to show that you indeed have made a pulse train. Make a higher resolution plot of the center peak by making a plot from -50 to +50 seconds. On this plot include a Gaussian peak with \( t_0 = 10 \) to show that the 31 harmonics are giving a reasonable representation of the peak. The Fourier series is a summation representing the spectrum at the bottom of slide 7.9-6.

(b 3pts) Consider a low pass filter with RC = 10 seconds. Use the gain equation on slide 8.3-9 to adjust the amplitudes of each frequency in your Fourier series from part (a). Since the gain equation is an even function you can expect the distortion to be even. Again make two plots, one from -150 to +150 seconds and the other from -50 to +50 seconds. Include the original and filtered waveform on the graph.

(c 3pts) Consider a low pass filter with RC = 10 seconds. Use the phase equation on slide 8.3-10 to adjust the phases of the Fourier series from part (a). This will introduce an interesting looking distortion that is neither odd nor even. Again make two plots, one from -150 to +150 seconds and the other from -50 to +50 seconds. Include the original and filtered waveform on the graph.

(d 4pts) Consider a low pass filter with RC = 10 seconds. Combine the distortions from parts (b) and (c). Again make two plots, one from -150 to +150 seconds and the other from -20 to +80 seconds. Include the original and filtered waveform on the graph.

This result was obtained by multiplication of the filter and signal spectra. Demonstrate the same result is obtained by convolution in the time domain. That is, convolve a Gaussian having characteristic parameter \( t_0 \) with a single-sided exponential having a decay constant RC. The two results should be identical. Show this in a separate graph.

(e 5pts) This part of the question explores the propagation of area, means and variance.

(1) Determine the area under the center peak for the pulse train in part (a) and compare it to the area under the filtered center peak in part (d). Hint: Solve the definite integrals numerically, paying close attention to the limits of integration. (2) Show that the mean of the center peak in part (d) is that from part (a) plus the mean of the RC filter temporal response. You will need to normalize each function to unit area by dividing the result from part (1). (3) Show that the variance of the center peak in part (d) is the sum of the variance from part (a) plus the variance of the RC filter. Note: Because peak distortion in part (d) stretches it close to the trailing peak, you can't set the limits of integration wide enough to get an answer accurate to many significant figures.

(f 10pts) Repeat part (d) using two ganged filters, both having RC = 10 seconds. The goal is to show the distortion added by the increased amount of filtering. Use the gain
expression on slide 8.4-4 to obtain the amplitude function. Follow the procedure given on slide 8.3-10, replacing \( \frac{X_c}{(R + X_c)} \) by its square. Convert the result to the form, \( a + jb \), then calculate the magnitude using \( (a^2 + b^2)^{1/2} \). Hint: I found it easier to substitute \( f_{\text{db}} = \frac{1}{2\pi RC} \) prior to the conversion since it reduced the number of symbols being manipulated.

You will need to develop your own phase equation. Once you get the ratio, \( \text{Im}/\text{Re} \), you will note that the real part presents a problem. That is, depending upon the value of \( f \) it can be positive (which is okay), zero (which causes Mathcad to fail with a divide by zero), and negative (which causes the arctangent function to return the incorrect angle). You will need to use the Mathcad if function to trap out each of these cases and handle them separately. For the case where the real part is negative you need to look at the complex plane to determine the correct phase angle and modify the output of the arctangent function. Please note that this is not a Mathcad failure, the arctangent itself can only generate angles in quadrants I and IV, you always have to do a sign test on the real part to correctly place angles in quadrants II and III.

Demonstrate the additional distortion by graphing the center peak for this part along with the center peak from part (d) and the unmodified Gaussian peak from part (a).

Comments on the phase ambiguity:
Consider the four vectors in the figure. Each is in a different quadrant. Ratios of the imaginary divided by the real for the four quadrants are:

<table>
<thead>
<tr>
<th>Quadrant</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>1/1</td>
</tr>
<tr>
<td>II</td>
<td>1/-1</td>
</tr>
<tr>
<td>III</td>
<td>-1/-1</td>
</tr>
<tr>
<td>IV</td>
<td>-1/1</td>
</tr>
</tbody>
</table>

As a result, the arctangent of the ratio cannot differentiate between a vector in quadrant I and III and quadrant II and IV. You need to examine the sign of the Real part to determine the correct quadrant, thus the correct phase angle.

(a 5pts) Develop a “brick wall” digital filter that has a rectangular bandpass from -0.2 to +0.2 Hz. Start with a filter length of 21 elements. Assume a temporal data spacing of 1 second, thus the Nyquist frequency is 0.5 Hz. Use the even waveform spectrum analyzer (from Exam 2, Question 4) to display the spectrum of the filter. For the graph, only display positive frequencies from 0 to 0.5 Hz. You will need to normalize the filter so that the sum of the filter elements is exactly 1. Comment on the shape of the filter spectrum, i.e what is the shape and what determines any characteristic parameter. What happens if the filter length is increased to a very large value, say 101 elements? Show the spectrum of the larger filter.

(b 3pts) Demonstrate the distortion that the brick wall filter of length 21 causes on a Gaussian peak having \( t_0 = 5 \). Make the data vector run from -50 to +50 seconds with the Gaussian centered at \( t = 0 \). For convolution use the equation shown on slide 9.4-6 as a template. Since your filter is already normalized, \( \text{norm} \) can be set to 1 in the convolution equation. For
your graph plot both the original Gaussian data and the filtered data. Distortion caused by the filter should be obvious.

(c 10pts) Take the brick wall filter from part (a) and get rid of the ringing by rounding the edges of its spectrum. Do this by thinking of a function which when multiplied times the sinc filter will round the rectangle edges by convolution. Keep the brick wall bandpass from -0.2 to +0.2 Hz, and a filter length of 21. The modified filter should have a spectrum with rounding from approximately -0.3 to -0.1 and 0.1 to 0.3 Hz. Graph the filter and its spectrum. What happens to the spectrum as the amount of rounding is decreased?

(d 2pts) Process the Gaussian peak of part (b) with the first filter developed in part (c). Use a graph to show that the ringing distortion has been eliminated.

(e 2pts) Make a new rounded brick wall filter with a rectangular bandpass from -0.1 to 0.1 Hz and rounding from approximately -0.2 to 0 Hz and 0 to +0.2 Hz. Process the Gaussian peak of part (b) with this filter and comment on any observed distortion.

(f 2pts) To see why there is so much distortion plot the spectrum of the Gaussian signal on the same graph as the spectrum of the filter. What signal frequencies are being most attenuated by the filter?

(g 5pts) Digital filters are most often used to decrease noise. This is accomplished by creating a filter having a variance less than 1. Determine the variance of the first filter in part (c). Do this in two ways (they give identical answers). First, use the propagation of variance for the multiplication and summing of one data point by the filter. In propagating variance assume the data variance (noise power) is 1. In the second method obtain the power spectrum of the filter by squaring the output of the even$(f, \text{filter})$ function. Integrate the power from 0.0 to 0.5 Hz and divide by the noise power ($1 \times 0.5$).

(h 3pts) Demonstrate that your result for part (g) is correct by creating a vector of 1,000 normally distributed random numbers have a mean of zero and a standard deviation of 1. The variance of the smoothed data should be very close to the value predicted in part (g).

(i 3pts) For the final part of this question add noise with a standard deviation of 0.05 to the Gaussian data peak in part (b). Process the data with the digital filter from part (c). On the same graph display the noise-free peak, the noisy peak, and the smoothed peak.

20 pts 4. There are two common methods to resolve ambiguities arising from aliasing – compare amplitudes with and without a lowpass RC filter; and, change the sampling frequency.

(a 10pts) A 2-kHz ADC generates frequencies at 100 Hz, 400 Hz and 800 Hz. The peak-to-peak voltages are 0.593 V @ 100 Hz, 0.672 V @ 400 Hz, and 0.988 V @ 800 Hz. An RC lowpass filter with $f_{3db} = 2$kHz is placed before the ADC. Now the peak-to-peak voltages are 0.185 V @ 100 Hz, 0.659 V @ 400 Hz, and 0.524 V @ 800 Hz. What are the actual frequencies? There are no frequencies higher than 7 kHz.

(b 10pts) A 2-kHz ADC generates frequencies at 79 Hz, 454 Hz, and 818 Hz. The sampling frequency is changed to 2.2 kHz. The frequencies now observed are 79 Hz, 254 Hz and 418 Hz. What are the actual frequencies? There are no frequencies higher than 6 kHz.

The exam will be due in class on Thursday April 24. Exams will be collected immediately after the lecture (not before nor during). If class ends early Alex will wait in
the room until 10:15.

This is an "open world" exam except for one condition. You may not talk to anyone about the exam except Prof. Lytle or Alex Davis. You may not share computer files or written worksheets. You may use all or parts of any worksheets made by Prof. Lytle and placed on the course web site. If you do this be sure state where it came from.

Because of the size of the class, you will only be able to seek help from Prof. Lytle or Alex once per day. Both of us will be available for help starting Monday morning April 21 and ending at 4:30 pm Wednesday, April 23. Professor Lytle will be available Wednesday, April 16. He will not be in the office from after class on Thursday, April 17 until Monday morning. Both of us will post office hours outside our doors.

Please be neat and organize your work. This usually helps when assigning partial credits. Sloppy graphs and unlabeled worksheets will lower your score. A typed test is preferred. When possible box your answer so it is obvious which response to grade.

All Mathcad worksheets have to be generated by version 12 or greater. Keep all files on a USB drive or CD. Do not keep files on a communal hard drive. Turn in all computer files with your exam.