8.3 Frequency-Dependent Impedance

- general idea for using frequency-dependent impedances
- ganged and band pass filters
- current flow through a resistor and a capacitor
- capacitive reactance
- gain and phase changes for a low pass filter
- attenuation and the Bode plot
- temporal response of the low pass filter
- connection to the Fourier transform
When the signal and noise spectra are different it is often possible to electronically discriminate against the noise. For Johnson noise, one can use the following passive circuit (has no amplifiers) with frequency-dependent impedances. The subscript L denotes "low" frequencies while the subscript H denotes "high" frequencies.

Wired this way, the circuit is called a low pass filter. The gain of the filter is given by the voltage divider equation, where $0 \leq G_L \leq 1$.

$$G_L(f) = \frac{e_{out}(f)}{e_{in}(f)} = \frac{Z_L(f)}{Z_{L+H}(f)}$$

Note that the impedances, thus the gain, are functions of frequency. Gain is usually a continuous function of frequency, i.e. it has no discontinuities.
A single-stage low pass filter is often sufficient to reduce Johnson noise to an acceptable level. In contrast, the gain of a single-stage filter is often insufficient to reduce a high frequency interference. To obtain a greater rejection at high frequencies, low pass filters can be ganged.

The gain of ganged filters is given by multiplication in the frequency domain,

\[ G_T = G_1 \cdot G_2 \quad \text{or} \quad A_T(\text{dB}) = A_1(\text{dB}) + A_2(\text{dB}) \]

Temporal distortion is given by double convolution with an exponential.

\[ V_{out}(t) = V_{in}(t) \otimes \exp(t^0) \otimes \exp(t^0) \]
Consider the rightmost voltage divider. If the signal is measured across $Z_H$ the divider is a high pass filter.

Its gain is given by the following expression.

$$G_2(f) = \frac{Z_{H2}(f)}{Z_{L2+H2}(f)}$$

The combination of a low pass and high pass filter yields a bandpass filter. This filter is useful for removing Johnson noise from a signal appearing within a narrow band of frequencies. The gain of the combination is again given by the product of the two gains, $G_T = G_1 \cdot G_2$ or $A_T(dB) = A_1(dB) + A_2(dB)$. 

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In anticipation of phase changes, consider a voltage written as a complex wave,

\[ e(t) = e_0 e^{j2\pi ft} \]

where \( e_0 \) is the amplitude, \( f \) is the frequency, and \( j \) is \( \sqrt{-1} \).

When a time-varying voltage is applied across a resistor, as shown in the figure, it induces a current according to Ohm's Law. The relationship between complex current and complex voltage is given by the circuit impedance, \( Z \).

For a circuit containing only resistance, the impedance is given by the resistance.

This last expression can be used to compute the complex current. The current is in phase with the voltage, and the ratio of the voltage and current amplitudes is independent of \( f \).
Current Flow through a Capacitor

What is the current when a time-varying voltage is applied across a capacitor? To answer this question we need to start with the definition of capacitance.

\[ C = \frac{Q}{e} \quad \text{or} \quad Q = Ce \]

A change in voltage \( \Delta e \) will then cause a change in charge \( \Delta Q \), where \( \Delta Q = C\Delta e \). Since current is defined as the change in charge with time, the \( \Delta \) terms can be replaced with differentials with respect to \( t \).

\[ i(t) = \frac{dQ}{dt} = C \frac{de(t)}{dt} = j2\pi f Ce_0 e^{j2\pi ft} \]

Note that the induced current is 90° out of phase with the voltage. This can be seen by remembering that \( j = e^{j\pi/2} \) and rewriting the equation.

\[ i(t) = 2\pi f Ce_0 e^{j2\pi ft} e^{j\pi/2} = 2\pi f Ce_0 e^{j(2\pi ft + \pi/2)} \]
Capacitive Reactance

For a capacitor, the time varying current and voltage are related by an equation similar to Ohm's Law,

\[ e(t) = X_C \cdot i(t) \]

where \( X_C \) is called the capacitive reactance.

\[ X_C = \frac{e_0 e^{j2\pi ft}}{j2\pi f \cdot C \cdot e^{j2\pi ft}} = \frac{1}{j2\pi f C} = \frac{-j}{2\pi f C} \]

The relationship between the current and voltage amplitudes is given by the impedance.

\[ Z = \sqrt{X_C X_C^*} = \sqrt{\frac{-j}{2\pi f C} \cdot \frac{j}{2\pi f C}} = \frac{1}{2\pi f C} \]

\[ i_0 = \frac{e_0}{1/2\pi f C} = 2\pi f Ce_0 \]

The impedance of the capacitor is infinity at zero frequency, and is zero at infinite frequency.
Consider the circuit at the right that consists of a series connected resistor and capacitor. When the output is measured across the capacitor the result is a frequency-dependent voltage divider.

The gain of the voltage divider is given by the following expression,

$$G_{\text{lowpass}}(f) = \frac{Z_C(f)}{Z_{R+C}(f)}$$

where $Z_{R+C}$ is the impedance of the resistor/capacitor combination. The value of $Z_{R+C}$ is given by the following.

$$Z_{R+C} = \sqrt{(R + X_C)(R + X_C)^*}$$

$$Z_{R+C} = \sqrt{(R - j/2\pi f C)(R + j/2\pi f C)}$$

$$Z_{R+C} = \sqrt{R^2 + 1/(2\pi f C)^2}$$
The values of $Z_C$ and $Z_{R+C}$ can be used to determine the gain of the low pass filter.

$$G_{\text{lowpass}}(f) = \frac{1/2\pi f C}{\sqrt{R^2 + 1/(2\pi f C)^2}} = \frac{1}{\sqrt{(2\pi RCf)^2 + 1}}$$

This expression can be simplified by defining a characteristic frequency, $f_{3\text{dB}} = 1/2\pi RC$.

$$G_{\text{lowpass}}(f) = \frac{1}{\sqrt{(f/f_{3\text{dB}})^2 + 1}}$$

Does the equation have the correct properties for a low pass filter? Yes, because when $f = 0$, $G = 1$; when $f = \infty$, $G = 0$; and, when $f = f_{3\text{dB}}$, $G = 1/\sqrt{2}$. 

RC = 1

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Phase Changes in an RC Filter

Since the impedance of an RC filter is complex, there will be changes in phase as a function of frequency. The magnitude of the phase change is obtained from the complex impedance.

\[
\frac{X_C}{R + X_C} = \frac{-j/2\pi f C}{R - j/2\pi f C} = \frac{(1/2\pi f C)^2}{R^2 + (1/2\pi f C)^2} - j \frac{R/2\pi f C}{R^2 + (1/2\pi f C)^2}
\]

The phase change is given by the angle in the complex plane.

\[
\theta = \arctan\left(\frac{\text{Im}}{\text{Re}}\right) = \arctan\left(-\frac{R/2\pi f C}{(1/2\pi f C)^2}\right) = \arctan\left(-2\pi f RC\right) = \arctan\left(-\frac{f}{f_{3dB}}\right)
\]

When \( f = 0 \), \( \theta = 0^\circ \); when \( f = \infty \), \( \theta = -90^\circ \); and, when \( f = f_{3dB} \), \( \theta = -45^\circ \). The phase changes most rapidly within the range, \( 0.1 f_{3dB} < f < 10 f_{3dB} \).
A Bode plot of an RC filter gives attenuation versus the log\( \left( \frac{f}{f_{3dB}} \right) \).

\[
A_{dB}(f) = 20 \log \left( \frac{1}{\sqrt{\left( \frac{f}{f_{3dB}} \right)^2 + 1}} \right) = -10 \log \left[ \left( \frac{f}{f_{3dB}} \right)^2 + 1 \right]
\]

If noise is spread evenly over all frequencies and the signal is restricted to frequencies close to zero, the gain of the RC filter will discriminate against noise.

Note that the graph of \( G \) versus \( f \) on slide 9 shows more clearly the signal distortion. Also note that the rapid phase change is restricted to two decades of frequency about \( f_{3dB} \).

(Bode is pronounced Boh\-dee in English and Boh\-duh in Dutch)
If $e_{in}$ is a square wave, the output has the sharp edges converted into an exponential rise and decay.

Since the filter multiplies in the frequency domain, it must be convolved with the signal in the time domain. The graph shows that the temporal behavior of the RC filter is a single-sided exponential. The decay constant is given by $RC$.

It is the convolution with the single-sided exponential that causes temporal signal distortion.

Does $RC$ have units of time?

$$e_{RC}(t) = e^{-\frac{t}{RC}}$$

$$RC \rightarrow \Omega \cdot F = \frac{V}{A} \cdot \frac{C}{V} = \frac{s}{C} = s$$
The Fourier transform of a single-sided exponential was previously shown to have the following amplitude (slide 7.10-6),

\[ e(f) = \frac{1}{2\pi} \frac{1}{\sqrt{(f^0/2)^2 + f^2}} \]

where \( f^0 = 1/\pi t^0 = 1/\pi RC \). Given, \( f_{3dB} = 1/2\pi RC \), \( f^0 \) can be rewritten as, \( f^0 = 2f_{3dB} \).

The Fourier transform result can be rewritten using \( f_{3dB} \). Since we know the RC filter gain is unity at \( f = 0 \), the gain can be written using a ratio of the Fourier transform results.

\[ G(f) = \frac{e(f)}{e(0)} = \frac{1}{2\pi} \frac{1}{\sqrt{f_{3dB}^2 + f^2}} \]

This result is identical to that derived using complex impedances.
The temporal signal is convolved with a single-sided exponential. The red line is for voltage, the blue line for power. The voltage decay constant is $RC$, the power decay constant is $RC/2$. In the graph $RC = 1$.

The signal spectrum in volts is multiplied by the square root of a Lorentzian (red). The spectrum in watts is multiplied by a Lorentzian (blue). In both cases $f_{3db}$ is given by $1/(2\pi RC)$. In the graph $f_{3db} = 0.16$.