8.5 Modulation of Signals

• basic idea and goals
• measuring atomic absorption without modulation
• measuring atomic absorption with modulation
• the tuned amplifier, diode rectifier and low pass filter
• the lock-in amplifier measurement
• mixer operation with the signal and interferences
• heterodyning
Often a signal naturally appears in a spectral region that contains significant amounts of noise.

- signal spectrum near \( f = 0 \) with significant \( 1/f \) noise
- signal spectrum near interferences, e.g. a 120 Hz power supply interference

If a signal can be modulated, it can be moved to a spectral region away from \( 1/f \) noise or interferences. There is an additional practical advantage. It is difficult to create a bandwidth less than 1 Hz near \( f = 0 \) Hz. In contrast, at 10 kHz inductors and capacitors can be combined to create bandwidths down to 0.01 Hz.

There are two important restrictions:

- The modulation must not simultaneously move the noise into the same spectral region as the signal. In practice this means that only the noise appearing after the modulation can be removed.
- Additionally, it is important that the noise be added to the signal. Multiplicative noise will not be removed.
Modulating a Signal Located Near DC

Modulation is achieved by multiplying the temporal signal by a cosine,

\[ F(t) \cdot \cos(2\pi t/t_{\text{mod}}) \leftrightarrow \Phi(f) \otimes \delta^+(f_{\text{mod}}) \]

where \( t_{\text{mod}} \) is much smaller than any temporal signal feature. For a signal with a Gaussian-shaped spectrum centered at 0 Hz and a 1 kHz modulation, the spectrum is the signal Gaussian duplicated and centered at ±1,000 Hz.

The blue arrow represents an interference at \( f=0 \) that arises after modulation. Thus it is not shifted to \( f_{\text{mod}} \).
A hypothetical instrument for measuring atomic absorption is shown at the right.

This instrument is susceptible to interference from flame emission and/or room light that might reach the detector (blue line). Thus the calculation of absorption becomes erroneous.

$$A = -\log\left(\frac{I + I_n}{I_0 + I_n}\right)$$

The instrument is also susceptible to $1/f$ noise. Any lamp drift can be removed by dividing the signal by the output of a reference detector, but that scheme does not remove other sources of $1/f$ noise.
A hypothetical atomic absorption instrument using modulation is shown below. The dashed red line is the modulated signal, while the single, solid red line is the un-modulated signal. The solid blue line parallel to the dashed line is the un-modulated interference.

The graph shows the signal plus interference prior to the bandpass filter. The interference remains until the bandpass filter and tuned amplifier. The tuned amplifier also rejects Johnson noise.
The left graph below shows the signal after the tuned amplifier, only the modulated portion remains. Note that the square wave is now a cosine - all the other harmonics were removed. The right graph shows the demodulated signal after the low pass filter.

The absorption calculation is done after the low pass filter, thus it is done only on the modulated/demodulated signal. This eliminates the error caused by the interference.

In practice the modulation is done at much higher frequencies than that shown on the graph!
A tuned amplifier can be created by placing a twin-T rejection filter in the feedback loop. When \( f = f_{\text{reject}} \) the impedance of the twin-T goes to a value near 1,000 M\( \Omega \).

The value of \( R_f \) controls the bandpass, while the value of \( R_f / R_{\text{in}} \) determines the gain of the amplifier at \( f_{\text{reject}} \). Note that \( f_{\text{reject}} \) is now the frequency passed.
The diode and low pass filter combination demodulate the ac signal.

In the graphs below the signal is shown at three points - incoming, after rectification, and after low pass filtering.
The Lock-In Amplifier

A hypothetical atomic absorption instrument using modulation with lock-in amplifier detection is shown at the right. The dotted green line indicates the components found within the lock-in amplifier. The major changes from the previous instrument are the inclusion of a reference waveform and the use of a mixer to effect multiplication.

The circuit diagram of a mixer is shown at the right. The signal, reference and output are all coupled through coils. The diode bridge performs the multiplication. The product signal appears across the two center-tapped coils.
Mixer Processing of the Signal

Let $f_m$ be the modulation frequency and $f_s$ be the signal frequency. Additionally, let the phase of the reference, with respect to the modulation, be $\phi$. The modulated signal is a cosine product, while the phase adjustable reference is a cosine.

$$S = A \cos(2\pi f_s t) \cos(2\pi f_m t) \quad R = \cos(2\pi f_m t + \phi)$$

The mixer output is the product of the two cosines.

$$M = R \cdot S = A \cos(2\pi f_s t) \cos(2\pi f_m t + \phi) \cos(2\pi f_m t)$$

$$M = A \cos(2\pi f_s t) \left[ \frac{1}{2} \cos(\phi) + \frac{1}{2} \cos(2\pi 2f_m t + \phi) \right]$$

$$M = \frac{A}{2} \cos(2\pi f_s t) \cos(\phi) + \frac{A}{2} \cos(2\pi f_s t) \cos(2\pi 2f_m t + \phi)$$

The low pass filter output removes the $2f_m$ component.

$$L = \frac{A}{2} \cos(2\pi f_s t) \cos(\phi)$$

For $\phi = 0^\circ$, $L = 0.5A$; for $\phi = 45^\circ$, $L = 0.35A$; and, for $\phi = 90^\circ$, $L = 0$. 

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Consider noise at the signal frequency which has a phase randomly varying with time. Let the phase of the reference be zero.

\[
N = \cos(2\pi f_m t)\cos(2\pi f_s t + \phi(t)) \quad R = \cos(2\pi f_m t)
\]

\[
M = N \cdot R = \frac{1}{2} \cos(2\pi f_s t + \phi(t)) + \frac{1}{2} \cos(2\pi 2 f_m t)\cos(2\pi f_s t + \phi(t))
\]

\[
L = \frac{1}{2} \int \cos(2\pi f_s t + \phi(t)) \, dt = 0
\]

The integration in the last expression is performed by the low pass filter response being convolved with the signal. Two requirements exist. First, the phase randomness has to be uniform over all phase angles (0°-360°). And second, the low pass filter time constant has to be sufficiently large so that the phase randomly takes all possible values.
An interference at the same frequency as the signal can be removed as long as the two cosines do not have the same phase.

Operationally, the interference is purposely connected to the lock-in input. The reference phase is then adjusted until $\phi = 90^\circ$. The signal plus interference is then connected to the lock-in input. The signal will appear at the output of the low pass filter with no contribution from the interference.

This procedure only works well when the signal and interference are 45-90° out of phase with each other. Also, the rejection is not total. Noise on the interference will be passed through the mixer and low pass filter with some small efficiency.
Heterodyning

Heterodyning is simply the multiplication of two frequencies to produce an output which has sum and difference frequencies. This is shown at the right.

The lock-in amplifier homodynes.

There are four primary reasons why heterodyning is used:
• Commercial lock-in amplifiers operate from 0.01 Hz to 150 kHz. Heterodyning is used to move higher frequencies into this range.
• The mixer is followed by an amplifier optimized for performance at a narrow-band, fixed frequency. The signal can be shifted to the optimum frequency by adjusting the local oscillator.
• The mixer can be used to generate new frequencies.
• A final use is shifting a high frequency by a very small amount. Thus, 100 MHz can be shifted using a 0 - 1,000 Hz local oscillator. The result is frequencies from 100,000,000 - 100,001,000 MHz. This precision would be difficult to obtain by any method other than using a frequency synthesizer.
For an infinitely long averaging time, the signal spectrum is multiplied by an impulse function. The temporal signal is then convolved with a unit amplitude cosine.

Averaging is accomplished in the time domain by varying the length and amplitude of the cosine. If the convolving cosine has a finite length obtained by truncation (multiplication by a rectangle), the impulses in the frequency domain will be replaced by sinc functions. Some lock-ins allow exponential averaging (multiplication of the cosine by an even exponential), which will produce Lorentzian functions in the frequency domain.