7.3 Even Basis Set Transforms

There are five even Fourier transform pairs that need to be *memorized*.

- impulse $\leftrightarrow$ constant
- cosine $\leftrightarrow$ even double impulse
- rectangle $\leftrightarrow$ sinc function
- Gaussian $\leftrightarrow$ Gaussian
- even exponential $\leftrightarrow$ Lorentzian function
An impulse function represents an infinitely narrow pulse occurring at a specified frequency or time. A temporal impulse will be denoted by $\delta(t^0)$. When $t = t^0$, $\delta(t^0) = 1$. When $t \neq t^0$, $\delta(t^0) = 0$.

A constant function represents a fixed amplitude over all time or frequencies. A frequency constant will be denoted by $C(f)$ and extends from $-\infty$ to $+\infty$.

The Fourier transform of an impulse function at zero time, $\delta(0)$, is a constant in the frequency domain, $C(f) = A$. 
Fourier transforms are independent of axis labels. This dramatically reduces the amount of memorization necessary!

The Fourier transform of an impulse function at zero frequency, $\delta(0)$, is a constant in the time domain, $C(t) = A$. 

\[
\begin{align*}
\delta(0) & \quad 2A \\
A & \quad A \\
-f & \quad (0,0) \\
C & = A \\
2A & \quad -t \\
\end{align*}
\]
The cosine is defined with a characteristic period, \( \cos(2\pi t/t^0) \), or characteristic frequency, \( \cos(2\pi f/f^0) \). Both extend from \(-\infty\) to \(+\infty\). The even double impulse, \( \delta\delta^+(t^0) \) or \( \delta\delta^+(f^0) \), is defined as two impulse functions with equal amplitudes of 0.5 occurring at two specified times, \( \pm t^0 \), or at two specified frequencies, \( \pm f^0 \).

\[
\delta\delta^+(t^0) = \frac{1}{2} \delta(t^0) + \frac{1}{2} \delta(-t^0)
\]

The Fourier transform of a temporal cosine is a double impulse in the frequency domain. Note that \( f^0 = 1/t^0 \).
The temporal rectangle function is defined as \( \text{rect}(t^0) = 1 \) for \(-t^0/2 \leq t \leq +t^0/2\), and \( \text{rect}(t^0) = 0 \) for all other values of \( t \). The frequency rectangle function has a similar definition with \( f \) substituted for \( t \). The frequency sinc function is defined as \( \text{sinc}(f^0) = \frac{\sin(\pi f/f^0)}{(\pi f/f^0)} \). The temporal sinc function has a similar definition with \( t \) substituted for \( f \). Note that \( \text{sinc}(0) = 1 \).

The Fourier transform of a temporal rectangle is a sinc function in the frequency domain. Note that \( f^0 = 1/t^0 \).
Gaussian and Gaussian

The temporal Gaussian function is defined as \( \exp[-\pi (t/t^0)^2] \). The frequency Gaussian is identical with \( f \) replacing \( t \), and \( f^0 \) replacing \( t^0 \). With this functional definition, the mean is zero, the standard deviation is \( t^0/(2\pi)^{1/2} \), and the area is equal to the pre-exponential factor times \( t^0 \).

The Fourier transform of a temporal Gaussian is a Gaussian in the frequency domain. Note that \( f^0 = 1/t^0 \).
Even Exponential and Lorentzian

The temporal even exponential is defined as $\exp(-|t|/t^0)$. Because of the absolute value operator, $t$ can range from $-\infty$ to $+\infty$. The frequency domain even exponential has a similar definition with $f$ substituted for $t$.

The frequency Lorentzian is defined as,

$$\text{lorenz}(f^0) = \frac{1}{\pi} \frac{f^0/2}{(f^0/2)^2 + f^2}$$

This definition requires that $f^0 = 1/(\pi t^0)$. Defined this way, $\Gamma = f^0$. The temporal Lorentzian is has a similar definition with $t$ substituted for $f$. 