Tuning the entanglement for a one-dimensional magnetic system with anisotropic coupling and impurities

Omar Osenda, Zhen Huang, and Sabre Kais
Department of Chemistry, Purdue University, West Lafayette, Indiana 47907, USA

We study a set of localized spins coupled through exchange interaction and subject to an external magnetic field. We demonstrate, for such a class of one-dimensional magnetic systems, that entanglement can be controlled and tuned by varying the anisotropy parameter in the Hamiltonian and by introducing impurities into the systems. In particular, for certain parameters, the entanglement is zero up to a critical point λ_c, where a quantum phase transition occurs, and is different from zero above λ_c.

Entanglement is a quantum-mechanical property that has no classical analog [1–4]. A pure state of a pair of quantum systems is called entangled if it is unfactorizable, for example, the singlet state of two-spin-$\frac{1}{2}$ particles, $1/\sqrt{2}\,(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$. A mixed state is entangled if it cannot be represented as a mixture of factorizable pure states [5–8]. Since in the seminal work of Einstein, Podolsky, and Rosen [9], there has been a quest for generating entanglement between quantum particles [2,10]. Investigation of quantum entanglement is currently a very active area and has been studied intensely due to its potential applications in quantum communications and information processing [2] such as quantum teleportation [11,12], superdense coding [13], quantum key distribution [14], telecloning [15], and decoherence in quantum computers [16,17].

Entanglement is the resource that enables quantum computation and quantum communication [18,19] and might lead to novel methods of measurement. Considerable effort has been devoted to the study of different sources of errors on quantum computing, mainly dealing with their effect on the quantum gates [20]. Therefore, it is of great interest to be able to tune it by varying parameters in the system. The feasibility of manipulating locally the exchange interaction [21] or the magnetic field [22] has been used in the literature to discuss different issues on the actual implementation of quantum computation [23]. Here, we demonstrate that the entanglement can be tuned in a class of one-dimensional magnetic systems by varying the anisotropy parameter and by introducing impurities into the systems. The Ising coupling is of great experimental importance, as it provides the basic quantum logic gate for many proposed implementations of quantum computing [20]. Although, naturally, greatest interest attaches to three-dimenstional lattices, the properties of one-dimensional magnetic systems with anistropic coupling and impurities are of both experimental and theoretical significance [24–27].

We consider a set of localized spin-$\frac{1}{2}$ particles coupled through exchange interaction J and subject to an external magnetic field of strength h. In the presence of impurities, the one-dimensional Hamiltonian is given by

$$H = -\frac{1+\gamma}{2}\sum_{i=1}^{N} J_{i,i+1}\sigma_i^x\sigma_{i+1}^x - \frac{1-\gamma}{2}\sum_{i=1}^{N} J_{i,i+1}\sigma_i^y\sigma_{i+1}^y$$
$$-\sum_{i=1}^{N} h\sigma_i^z, \tag{1}$$

where $J_{i,i+1}=J$ for all sites except the sites nearest to the impurity site located at i_m, $J_{i_m,i_m+1}=J_{i_m-1,i_m}=J(1+\alpha)$, α measures the strength of the impurity which is located at site i_m, σ^a are the Pauli matrices ($a=x,y,z$), and N is the number of sites. We assume periodic boundary conditions. For this model, it is convenient to define a dimensionless coupling constant $\lambda=J/2h$. γ is an anisotropy parameter, for $\gamma=1$ Eq. (1) reduces to the Ising model, whereas for $\gamma=0$ it is the XY model. For all the interval $0<\gamma\leq 1$ the models belong to the Ising universality class, for $N=\infty$ they undergo a quantum phase transition at the critical value $\lambda_c=1$, where the correlation length [28] ξ diverges as $\xi\simeq|\lambda-\lambda_c|^{-1}$.

For the pure case, $\alpha=0$ and $J_{i,j}=J$, Osterloh *et al.* [28] examined the entanglement between two spins, of position i and j, in the chain as the system goes through quantum phase transition. They demonstrate that entanglement shows scaling behavior in the vicinity of the transition point. Moreover, they verified the universality in the critical region of the entanglement by considering the properties of the family of models for $0<\gamma\leq 1$.

All the information needed for quantifying the entanglement in this case is contained in the reduced density matrix $\rho(i,j)$. Wootters [7] has shown, for a pair of binary qubits, that the concurrence C, which goes from 0 to 1, can be taken as a measure of entanglement. The concurrence between sites i and j is defined as [7]

$$C(\rho) = \max\{0, \epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4\}, \tag{2}$$

where the ϵ_i's are the eigenvalues of the Hermitian matrix $R \equiv (\sqrt{\rho}\,\tilde{\rho}\,\sqrt{\rho})^{1/2}$. The spin-flipped state $\tilde{\rho}$, for a general state ρ of two qubits, is given by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{3}$$
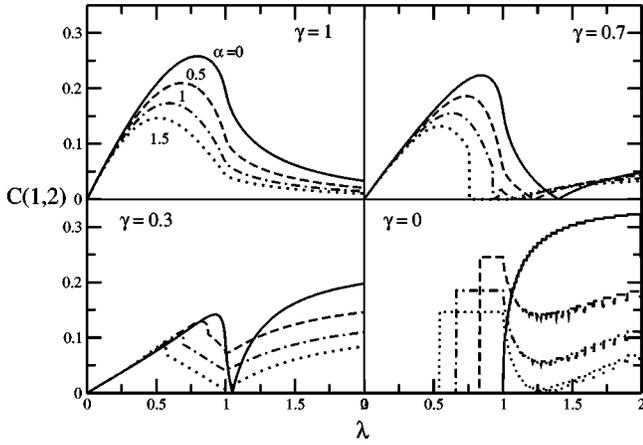
FIG. 1. The nearest-neighbor concurrence $C(1,2)$ for different values of the anisotropy parameter $\gamma = 1, 0.7, 0.3, 0$ with an impurity located at $i_m = 3$ as a function of the reduced coupling constant $\lambda = J/2h$, where J is the exchange interaction constant and h is the strength of the external magnetic field. The curves correspond to different values of the impurity strength $\alpha = 0, 0.5, 1, 1.5$ with system size $N = 201$.

FIG. 3. The next-nearest-neighbor concurrence $C(i, i+2)$ for both the Ising model and the XY model for $\alpha = 0.3$. The different curves show the location of the pair relative to the location of the impurity, which is located at position i_m.

where the ρ^* is the complex conjugate of ρ and is taken in the standard basis [7], which for a pair of spin-$\frac{1}{2}$ particles is $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

The structure of the reduced density matrix follows from the symmetry properties of the Hamiltonian. However, for this case the concurrence $C(i,j)$ depends on i, j, and i_m and not only on the difference $|i-j|$ as for the pure case. The matrix elements of the reduced density matrix needed for calculating the concurrence $C(i,j)$ are obtained numerically using the formalism developed by Lieb *et al.* [29].

First, we examine the change of the entanglement for the Ising model ($\gamma = 1$) for different values of the impurity strength α as the parameter λ, which induces the quantum phase transitions, varies. Figure 1 shows the change of the nearest-neighbor concurrence $C(1,2)$ with the impurity located at $i_m = 3$ as a function of λ for different values of α.
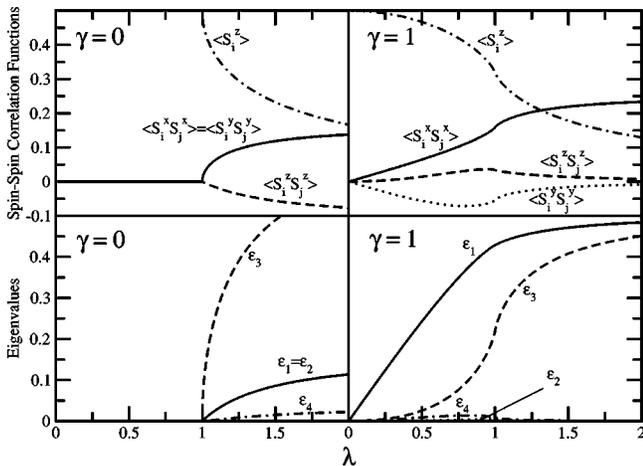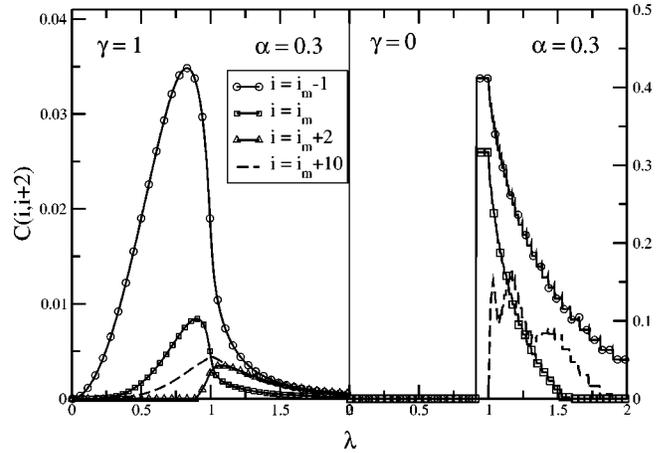
One can see clearly in Fig. 1 that the entanglement can be tuned down by increasing the value of the parameter α. For $\alpha = 1.5$, the concurrence approaches zero above the critical point $\lambda_c = 1$. The system size was taken $N = 201$ based on finite-size scaling analysis. Analysis of all the results for the pure case ($\alpha = 0$) for different system size ranging from $N = 41$ up to $N = 401$ collapse onto a single curve [28]. Thus, all key ingredients of the finite-size scaling are present in the concurrence. This holds true for the impurity problem as long as we consider the behavior of the value of λ for which the derivative of the concurrence attains its minimum value versus the system size. As expected there is no divergence of the derivative $dC(1,2)/d\lambda$ for finite N, but there are clear anomalies. By examining $\ln(\lambda_c - \lambda_m)$ versus $\ln N$ for $\alpha = 0.1$, one obtains that the minimum λ_m scales as $\lambda_m \sim \lambda_c + N^{-0.93}$ and $dC(1,2)/d\lambda$ diverges logarithmically with increasing system size. For a system with the impurity located at larger distance $i_m = 10$ and the same $\alpha = 0.1$, $\lambda_m \sim \lambda_c + N^{-0.85}$, showing that the scaling behavior depends on the

FIG. 2. The spin-spin correlation functions and the eigenvalues ϵ_i, see Eq. (2) in the text, for the XY model ($\gamma = 0$) and the Ising model ($\gamma = 1$) as a function of λ for $\alpha = 0$.
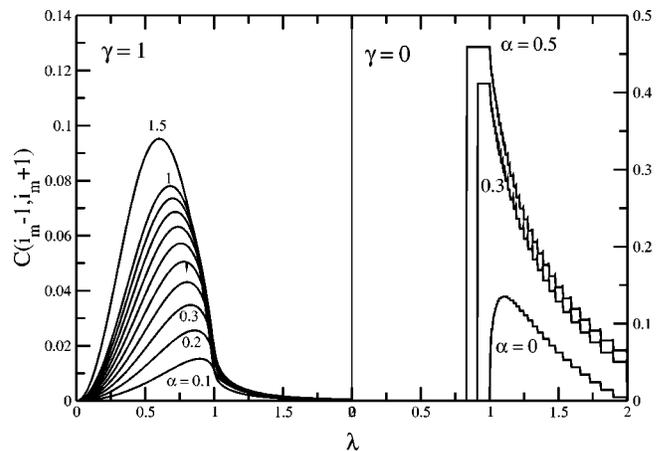

FIG. 4. The next-nearest-neighbor concurrence $C(i_m-1, i_m+1)$ as a function of λ for different values of α for the Ising model and the XY model.

distance between the impurity and the pair of sites under consideration.

Figure 1 also shows the variation of nearest-neighbor concurrence as the anisotropy parameter γ decreases. For the XY model ($\gamma=0$), the concurrence for $\alpha=0$ is zero up to the critical point $\lambda_c=1$ and different from zero above $\lambda_c=1$. However, as α increases the concurrence develops a steps and the results strongly depend on the system size. For small system size, such as $N=101$, the steps and oscillations are large but become smaller as the system size increases as shown in Fig. 1 for $N=201$. But they disappear in the limit $N\rightarrow\infty$.

To examine the different behavior of the concurrence for the Ising model and the XY model, we took the system size to be infinite, $N\rightarrow\infty$, where the two models have exact solutions. However, the behavior is the same for a finite system with $N=201$. Figure 2 shows the behavior of the different spin-spin correlation functions and the eigenvalues ϵ_i as λ varies for the pure case $\alpha=0$. For the XY model, all the spin-spin correlation functions and the eigenvalues in Eq. (2) are zero up to the critical point and different above the critical point $\lambda_c=1$. For the Ising model, the spin-spin correlation functions and the four eigenvalues are different from zero for all values of λ as shown in Fig. 2. However, for the Ising model, the two leading eigenvalues, ϵ_1 and ϵ_3 approaches each other above $\lambda_c=1$ and the concurrence approaches zero as shown in Fig. 1.

Up to now we examined the nearest-neighbor concurrence $C(1,2)$ with an impurity located at $i_m=3$. Of course, for larger values of i_m the concurrence gets larger and approaches its maximum value, the pure case with $\alpha=0$, at large values $i_m\gg1$. It is worth mentioning that for the Ising model, the range of entanglement [30], which is the maximum distance between spins at which the concurrence is different from zero, vanishes unless the two sites are at most next-nearest neighbors. For $\gamma\neq1$, the range of entanglement is not universal and tends to infinity as γ tends to zero.

To examine the effect of the location of the impurity on the next-nearest-neighbor concurrence $C(i,i+2)$, we show in Fig. 3 the results for $i=i_m-1$, $i=i_m$, $i=i_m+2$, and $i=i_m+10$ for both models, the Ising and the XY for impurity strength $\alpha=0.3$. For $\gamma=1$, the next-nearest-neighbor concurrence tunned down as the pair $(i,i+2)$ is farther away from the location of the impurity. However, for $\gamma=0$ the $C(i,i+2)$ is zero up to the critical λ_c and it differs from zero for $\lambda>\lambda_c$ as shown in Fig. 3.

A similar analysis can be carried out for the next nearest-neighbor concurrence $C(i,i+2)$ as the impurity strength α varies. Figure 4 shows the next-nearest-neighbor concurrence as a function of λ when the impurity is located between the two sites, $C(i_m-1,i_m+1)$. For the Ising model, the maximum value of the concurrence increase as the value
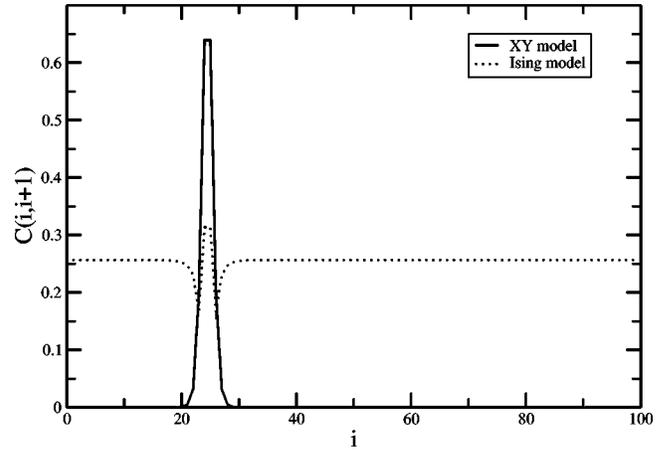

FIG. 5. The nearest-neighbor concurrence $C(i,i+1)$ for all the sites on the chain. $N=101$, $\alpha=0.9$, $\lambda=0.75$, and the impurity is located at $i_m=25$.

of α increases. However, the concurrence is zero above the critical point $\lambda_c=1$ as shown in the left side of Fig. 4. For the XY model, the picture is different. The concurrence $C(i_m-1,i_m+1)$ is zero up to the critical point and is different from zero above it. Again the appearance of small steps is related to the size of the system, they disappear in the infinite size limit.

To address the question what happens to the other spins in the system, we show in Fig. 5 the nearest-neighbor concurrence $C(i,i+1)$ for all the sites on the chain for a given $\alpha=0.9$ and $\lambda=0.75$. For the XY model with one impurity located at $i_m=25$, the entanglement is different from zero only in a small region where the pure system has identically zero entanglement between pairs of nearest- and next-nearest-neighbors spins. This fact allows for an almost local control of the entanglement: only few sites become entangled with their nearest and next-nearest neighbors as shown in Fig. 5. For the Ising model, the concurrence (and hence the entanglement) is rather insensitive to small changes in the exchange interaction. The entanglement between nearest neighbors tends to be reduced in the presence of impurity, while the entanglement between next-nearest-neighbors increases.

In summary, we have shown how entanglement can be tunned by changing the anisotropy parameter γ by going from the Ising model ($\gamma=1$) to the XY model ($\gamma=0$). For the XY model, the entanglement is zero up to the critical point λ_c, and is different from zero above λ_c. Moreover, by introducing impurities, the entanglement can be tunned down as the strength of the impurity α increases.

[1] C.H. Bennett and D.P. DiVincenzo, Nature (London) **404**, 247 (2000).

[2] C. Macchiavello, G.M. Palma, and A. Zeilinger, *Quantum*

Computation and Quantum Information Theory (World Scientific, Singapore, 2000).

[3] M. Nielsen and I. Chuang, *Quantum Computation and Quan-*

tum Communication (Cambridge University Press, Cambridge, 2000).

[4] J. Gruska, *Quantum Computing* (McGraw-Hill, New York, 1999).

[5] V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. **78**, 2275 (1997).

[6] S. Hill and W.K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[7] W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[8] G. Vidal, W. Dur, and J.I. Cirac, Phys. Rev. Lett. **89**, 027901 (2002).

[9] A. Einstein, B. Podlosky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[10] R. Blatt, Nature (London) **404**, 231 (2000).

[11] C.H. Bennet *et al.*, Phys. Rev. Lett. **70**, 1895 (1993).

[12] D. Bouwmeester *et al.*, Nature (London) **390**, 575 (1997).

[13] C.H. Bennet and S.J. Wiesener, Phys. Rev. Lett. **69**, 2881 (1992).

[14] A.K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[15] M. Murao, D. Jonathan, M.B. Plenio, and V. Vedral, Phys. Rev. A **59**, 156 (1999).

[16] D.P. DiVincenzo, Science **270**, 255 (1995).

[17] D. Bacon, J. Kempe, D.A. Lidar, and K.B. Whaley, Phys. Rev. Lett. **85**, 1758 (2000).

[18] C. Monroe, Nature (London) **416**, 238 (2002).

[19] W. Barnes, Nature (London) **418**, 281 (2002).

[20] Jonathan A. Jones, Phys. Rev. A **67**, 012317 (2003).

[21] Simon C. Benjamin, Phys. Rev. Lett. **2002**, 017904 (2002).

[22] D.P. DiVicenzo *et al.*, Nature (London) **408**, 339 (2000).

[23] F. Meier, J. Levy, and Daniel Loss, Phys. Rev. Lett. **90**, 047901 (2003).

[24] P.R. Hammar *et al.*, Phys. Rev. B **59**, 1008 (1999).

[25] M.C. Arnesen, S. Bose, and V. Vedral, Phys. Rev. Lett. **87**, 017901 (2001).

[26] X. Wang, Phys. Rev. A **64**, 012313 (2001).

[27] A. Taye, D. Michel, and J. Petersson, Phys. Rev. B **66**, 174102 (2002).

[28] A. Osterloh, L. Amico, G. Falci, and R. Fazio, Nature (London) **416**, 608 (2002).

[29] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. **60**, 407 (1961).

[30] D. Aharonov, Phys. Rev. A **62**, 062311 (2000).