Entanglement dynamics of one-dimensional driven spin systems in time-varying magnetic fields

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We study the dynamics of nearest-neighbor entanglement for a one-dimensional spin chain with a nearest-neighbor time-dependent Heisenberg coupling $J(t)$ between the spins in the presence of a time-dependent external magnetic field $h(t)$ at zero and finite temperatures. We consider different forms of time dependence for the coupling and magnetic field: exponential, hyperbolic, and periodic. Solving the system numerically, we examined the system-size effect on the entanglement asymptotic value. It was found that, for a small system size, the entanglement starts to fluctuate within a short period of time after applying the time-dependent coupling. The period of time increases as the system size increases and disappears completely as the size goes to infinity. Testing the effect of the transition constant for an exponential or hyperbolic coupling showed a direct impact on the asymptotic value of the entanglement; the larger the constant is, the lower the asymptotic value and the more rapid decay of entanglement are, which confirms the nonergodic character of the system. We also found that, when $J(t)$ is periodic, the entanglement shows a periodic behavior with the same period, which disappears upon applying periodic magnetic field with the same frequency. Solving the case $J(t) = \lambda h(t)$, for constant λ, exactly, we showed that the time evolution and asymptotic value of entanglement are dictated solely by the parameter $\lambda = J/h$ rather than the individual values of J and h, not only when they are time independent and at zero temperature, but also when they are time dependent but proportional at zero and finite temperatures for all degrees of anisotropy.

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I. INTRODUCTION

Quantum entanglement represents one of the cornerstones of the quantum mechanics theory with no classical analog [1]. Quantum entanglement is a nonlocal correlation between two (or more) quantum systems such that the description of their states has to be done with reference to each other even if they are spatially well separated. Understanding and quantifying entanglement may provide an answer for many questions regarding the behavior of the many-body quantum systems. Particularly, entanglement is considered as the physical property responsible for the long-range quantum correlations accompanying a quantum phase transition in many-body systems at zero temperature [2–4]. Entanglement plays a crucial role in many fields of modern physics, particularly, quantum teleportation, quantum cryptography, and quantum computing [5,6]. It is considered as the physical basis for manipulating linear superpositions of quantum states to implement the different proposed quantum-computing algorithms. Different physical systems have been proposed as promising candidates for the future quantum-computing technology [7–15]. It is a major task in each one of these considered systems to find a controllable mechanism to form and coherently manipulate the entanglement between a two-qubit system, creating an efficient quantum-computing gate. The coherent manipulation of entangled states has been observed in different systems such as isolated trapped ions [16], superconducting junctions [17], and coupled quantum dots, where the coupling mechanism in the latter system is the Heisenberg exchange interaction between electron spins [18–20]. One of the most interesting proposals for creating a controllable mechanism in coupled quantum-dot systems was introduced by Loss *et al*. [21,22]. The coupling mechanism is a time-dependent exchange interaction between the two valence spins on a doubled quantum-dot system, which can be pulsed over definite intervals resulting in a swap gate. This control can be achieved by raising and lowering the potential barrier between the two dots through controllable gate voltage. In a previous work, a two-atom system with time-dependent coupling was studied and the critical dependence of the entanglement and variance squeezing on the strength and frequency of the coupling was demonstrated [23].

Quantifying entanglement in the quantum states of multiparticle systems is in the focus of interest in the field of quantum information. However, quantum entanglement is very fragile due to the induced decoherence caused by the inevitable coupling to the environment. Decoherence is considered as one of the main obstacles toward realizing an effective quantum-computing system [24]. The main effect of decoherence is to randomize the relative phases of the possible states of the considered system. Quantum error correction [25] and decoherence-free subspace [26,27] have been proposed to protect the quantum property during the computation process. Nevertheless, offering a potentially ideal protection against environmentally induced decoherence is a difficult task. Moreover, a spin-pair entanglement is a reasonable measure for decoherence between the considered two-spin system and its environment constituted by the rest of the spins on the chain. The coupling between the system and its environment leads to decoherence in the system and sweeping out entanglement between the two spins. Therefore,

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monitoring the entanglement dynamics in the considered system helps us to understand the behavior of the decoherence between the considered two spins and their environment. Particularly, the effect of the environment size on the coherence of quantum states of the system can be considered by watching the spin-pair entanglement evolution versus the the number of sites N in the chain.

Developing new experimental techniques enabled the generation and control of multiparticle entanglement [28–33] as well as the fabrication of one-dimensional spin chains [34–36]. This progress in the experimental arena sparked an intensive theoretical research over the multiparticle systems and particularly the one-dimensional spin chains [37–46]. The dynamics of entanglement in XY and Ising spin chains has been studied considering a constant nearest-neighbor exchange interaction, in the presence of a time-varying magnetic field represented by step, exponential, and sinusoidal functions of time [47,48]. Furthermore, the dynamics of entanglement in a one-dimensional Ising spin chain at zero temperature was investigated numerically where the number of spins was seven at most [49]. The generation and transportation of the entanglement through the chain, which was irradiated by a weak resonant field under the effect of an external magnetic field, were investigated. Recently, the entanglement in an anisotropic XY model with a small number of spins, with a time-dependent nearest-neighbor coupling at zero temperature, was studied too [50]. The time-dependent spin-spin coupling was represented by a dc part and a sinusoidal ac part. It was observed that there is an entanglement resonance through the chain whenever the ac coupling frequency is matching the Zeeman splitting. Very recently, we have studied the time evolution of entanglement in a one-dimensional spin chain in the presence of a time-dependent magnetic field $h(t)$ considering a time-dependent coupling parameter $J(t)$, where both $h(t)$ and $J(t)$ were assumed to be of a step-function form [51]. Solving the problem exactly, we found that the system undergoes a nonergodic behavior. At zero temperature, we found that the asymptotic value of the entanglement depends only on the ratio $\lambda = J/h$. However, at nonzero temperatures, it depends on the individual values of h and J. Also, we have demonstrated that the quantum effects dominate within certain regions of the temperature-λ space, which vary significantly depending on the degree of the anisotropy of the system.

In this paper, we investigate the time evolution of quantum entanglement in a one-dimensional XY spin-chain system coupled through nearest-neighbor interaction under the effect of an external magnetic field at zero and finite temperatures. We consider both time-dependent nearest-neighbor Heisenberg coupling $J(t)$ between the spins on the chain and magnetic field $h(t)$, where the function forms are exponential, periodic, and hyperbolic in time. Particularly, we focus on the concurrence as a measure of entanglement between any two adjacent spins on the chain and its dynamical behavior under the effect of the time-dependent coupling and magnetic fields. We apply both analytical and numerical approaches to tackle the problem. We show that the time evolution and asymptotic behavior of entanglement depend only on the parameter $\lambda = J/h$ rather than the individual values of the coupling and the magnetic field, not only when they are time independent and at zero temperature, as was demonstrated by previous works [47,48,51], but also when they are time dependent but proportional at both zero and finite temperatures considering all degrees of anisotropy. As a consequence, we show that applying a proportional oscillatory magnetic field and coupling yields a nonoscillatory concurrence. Also, we demonstrate how the transition rate of an applied magnetic field or coupling, of exponential or hyperbolic forms, significantly affects the asymptotic value of the concurrence confirming the nonergodic behavior of the system. We also test the effect of the spin-chain size on the fluctuations in the system dynamics and the asymptotic value of the entanglement.

This paper is organized as follows. In Sec. II, we present our model and discuss the numerical solution for the the XY spin chain for a general form of the coupling and magnetic field. Then, we present an exact solution for the system for the special case $J(t) = \lambda h(t)$, where λ is a constant. In Sec. III, we evaluate the entanglement using the magnetization and the spin-spin correlation functions of the system. We present our results and discuss them in Sec. IV. Finally, in Sec. V, we conclude and discuss future directions.

II. TIME-DEPENDENT XY MODEL

A. Numerical solution

In this section, we present a numerical solution for the XY model of a spin chain with N sites in the presence of a time-dependent external magnetic field $h(t)$. We consider a time-dependent coupling $J(t)$ between the nearest-neighbor spins on the chain. The Hamiltonian for such a system is given by

$$H = -\frac{J(t)}{2}(1+\gamma)\sum_{i=1}^{N}\sigma_i^x\sigma_{i+1}^x - \frac{J(t)}{2}(1-\gamma) \times \sum_{i=1}^{N}\sigma_i^y\sigma_{i+1}^y - \sum_{i=1}^{N}h(t)\sigma_i^z, \tag{1}$$

where σ_i's are the Pauli matrices and γ is the anisotropy parameter. For simplicity, we consider $\hbar = 1$ throughout this paper. We define the raising and lowering operators $a_i^\dagger, a_i$ as

$$a_i^\dagger = \tfrac{1}{2}\left(\sigma_i^x + i\sigma_i^y\right), \quad a_i = \tfrac{1}{2}\left(\sigma_i^x - i\sigma_i^y\right). \tag{2}$$

Following the standard procedure to treat the Hamiltonian (1), we introduce Fermi operators $b_i^\dagger, b_i$ [52]:

$$a_i^\dagger = b_i^\dagger \exp\left(i\pi\sum_{j=1}^{i-1}b_j^\dagger b_j\right), \quad a_i = \exp\left(-i\pi\sum_{j=1}^{i-1}b_j^\dagger b_j\right)b_i. \tag{3}$$

Then, by applying Fourier transformation, we obtain

$$b_i^\dagger = \frac{1}{\sqrt{N}}\sum_{p=-N/2}^{N/2}e^{ij\phi_p}c_p^\dagger, \quad b_i = \frac{1}{\sqrt{N}}\sum_{p=-N/2}^{N/2}e^{-ij\phi_p}c_p, \tag{4}$$

where $\phi_p = \frac{2\pi p}{N}$. Therefore, the Hamiltonian can be written as

$$H = \sum_{p=1}^{N/2}\tilde{H}_p, \tag{5}$$

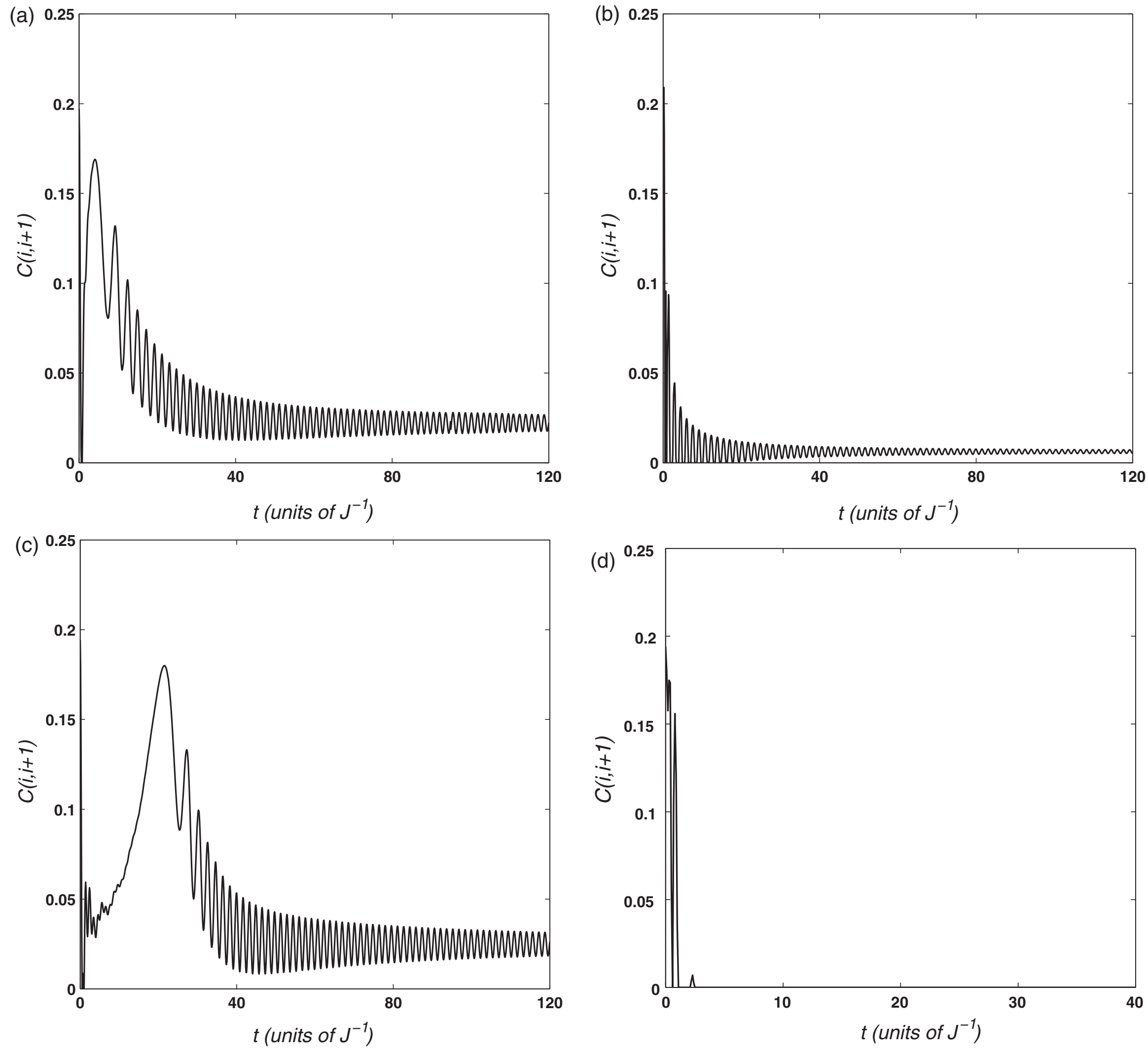

FIG. 1. $C(i,i+1)$ as a function of t with $J_0 = 0.5,\ J_1 = 2,\ h = 1,\ N = 1000$ at $kT = 0$ and (a) $J = J_{\rm exp},\ K = 0.1$; (b) $J = J_{\rm exp},\ K = 10$; (c) $J = J_{\rm tanh},\ K = 0.1$; and (d) $J = J_{\rm tanh},\ K = 10$.

with $\tilde{H}_p$ given by

$$\tilde{H}_p = \alpha_p(t)[c_p^\dagger c_p + c_{-p}^\dagger c_{-p}] + iJ(t)\delta_p \times [c_p^\dagger c_{-p}^\dagger + c_p c_{-p}] + 2h(t), \tag{6}$$

where $\alpha_p(t) = -2J(t)\cos\phi_p - 2h(t)$ and $\delta_p = 2\gamma\sin\phi_p$.

As $[\tilde{H}_l,\tilde{H}_m] = 0$ for $l,m = 0,1,2,\ldots,N/2$, the Hamiltonian in the 2^N-dimensional Hilbert space can be decomposed into $N/2$ noncommuting sub-Hamiltonians, each in a four-dimensional independent subspace. By using the basis $\{|0\rangle, c_p^\dagger c_{-p}^\dagger|0\rangle, c_p^\dagger|0\rangle, c_{-p}^\dagger|0\rangle\}$, we obtain the matrix representation of $\tilde{H}_p$:

$$\tilde{H}_p = \begin{pmatrix} 2h(t) & -iJ(t)\delta_p & 0 & 0 \\ iJ(t)\delta_p & -4J(t)\cos\phi_p - 2h(t) & 0 & 0 \\ 0 & 0 & -2J(t)\cos\phi_p & 0 \\ 0 & 0 & 0 & -2J(t)\cos\phi_p \end{pmatrix}. \tag{7}$$

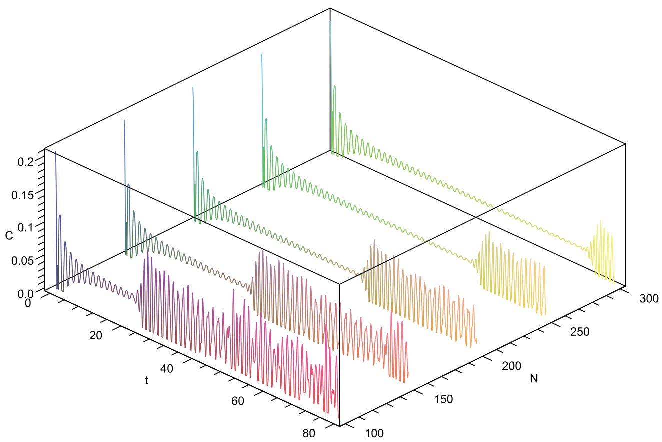

FIG. 2. (Color online) $C(i,i+1)$ as a function of t (units of J^{-1}) with $J = J_{\exp}$, $J_0 = 0.5$, $J_1 = 2$, $h = 1$, $K = 1000$ at $kT = 0$ and N varies from 100 to 300.

Initially, the system is assumed to be in a thermal equilibrium state and therefore its initial density matrix is given by

$$\rho_p(0) = \frac{e^{-\beta \tilde{H}_p(0)}}{Z}, \qquad Z = \mathrm{Tr}(e^{-\beta \tilde{H}_p(0)}), \tag{8}$$

where $\beta = 1/kT$, k is Boltzmann constant, and T is the temperature.

Since the Hamiltonian is decomposable, we can find the density matrix at any time t, $\rho_p(t)$, for the pth subspace by solving the Liouville equation, in the Heisenberg representation, given by

$$i\dot{\rho}_p(t) = [\tilde{H}_p(t), \rho_p(t)], \tag{9}$$

which gives

$$\rho_p(t) = U_p(t)\rho_p(0)U_p^{\dagger}(t), \tag{10}$$

where $U_p(t)$ is time evolution matrix, which can be obtained by solving the equation

$$i\,\dot{U}_p(t) = U_p(t)\tilde{H}_p(t). \tag{11}$$

To study the effect of a time-varying coupling parameter $J(t)$, we consider the following forms:

$$J_{\exp}(t) = J_1 + (J_0 - J_1)\,e^{-Kt}, \tag{12}$$

$$J_{\cos}(t) = J_0 - J_0 \cos(Kt), \tag{13}$$

$$J_{\sin}(t) = J_0 - J_0 \sin(Kt), \tag{14}$$

$$J_{\tanh}(t) = J_0 + \frac{J_1 - J_0}{2}\left\{\tanh\left[K\left(t - \frac{5}{2}\right)\right] + 1\right\}. \tag{15}$$

Note that Eq. (11) gives two systems of coupled differential equations with variable coefficients. Such systems can only be solved numerically, which we adopt in this paper. The results here are valid for any N (small or large), while in the thermodynamic limit $N \to \infty$, the sum over ϕ_p becomes an integral over ϕ, which we do not consider here [51,53]. The finite-size N case has been studied and treated exactly in different works, which lead to parity effects (see for instance Refs. [52–54]).

B. An exact solution for proportional J and h

In this section, we present an exact solution of the system using a general time-dependent coupling $J(t)$ and a magnetic field with the following form:

$$J(t) = \lambda\, h(t), \tag{16}$$

where λ is a constant. By using Eqs. (7), (11), and (16), we obtain

$$i\begin{pmatrix} \dot{u}_{11} & \dot{u}_{12} \\ \dot{u}_{21} & \dot{u}_{22} \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\begin{pmatrix} \frac{2}{\lambda} & -i\delta_p \\ i\delta_p & -4\cos\phi_p - \frac{2}{\lambda} \end{pmatrix} J(t) \tag{17}$$

and

$$i\,\dot{u}_{33} = -2\,\cos\phi_p\; J(t)\, u_{33}, \quad u_{44} = u_{33}. \tag{18}$$

Equation (17) can be rewritten as

$$i\,\dot{u}_j = J(t)\, H' u_j \tag{19}$$

for $j = 1,2$, where

$$H' = \begin{pmatrix} \frac{2}{\lambda} & i\delta_p \\ -i\delta_p & 4\cos\phi_p - \frac{2}{\lambda} \end{pmatrix}, \quad u_j = \begin{pmatrix} u_{j1} \\ u_{j2} \end{pmatrix}. \tag{20}$$

Introducing a unitary rotation matrix

$$S = \begin{pmatrix} \cos\theta & e^{i\phi}\sin\theta \\ -e^{-i\phi}\sin\theta & \cos\theta \end{pmatrix} \tag{21}$$

and using S to diagonalize H', we obtain

$$SH'S^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \tag{22}$$

where the angles ϕ and θ were found to be

$$\phi = (n+1)\pi, \quad \tan 2\theta = \frac{\delta_p}{2\cos\phi_p + \frac{2}{\lambda}}, \tag{23}$$

where $n = 0, \pm 1, \pm 2, \ldots$; therefore,

$$\sin 2\theta = \frac{\delta_p}{\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)}},$$

$$\cos 2\theta = \frac{2\cos\phi_p + \frac{2}{\lambda}}{\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)}}. \tag{24}$$

Finding λ_1 and λ_2, we get

$$\lambda_1 = \sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)} - 2\cos\phi_p,$$

$$\lambda_2 = -\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)} - 2\cos\phi_p. \tag{25}$$

Now, we define $v_j = Su_j$ and substituting in Eq. (19) we get

$$i\,\dot{v}_j = J(t)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} v_j. \tag{26}$$

Solving the last equation, we obtain

$$v_1 = \begin{pmatrix} \cos\theta\, e^{-i\lambda_1 \int_0^t J(t')dt'} \\ i\sin\theta\, e^{-i\lambda_2 \int_0^t J(t')dt'} \end{pmatrix}, \tag{27}$$

$$v_2 = \begin{pmatrix} i\sin\theta\, e^{-i\lambda_1 \int_0^t J(t')dt'} \\ \cos\theta\, e^{-i\lambda_2 \int_0^t J(t')dt'} \end{pmatrix}.$$

Finally, u is given by

$$u_{11} = \cos^2\theta e^{-i\lambda_1 \int_0^t J(t')dt'} + \sin^2\theta e^{-i\lambda_2 \int_0^t J(t')dt'}, \tag{28}$$

$$u_{12} = -i\sin\theta\cos\theta\left\{e^{-i\lambda_1 \int_0^t J(t')dt'} - e^{-i\lambda_2 \int_0^t J(t')dt'}\right\}, \tag{29}$$

$$u_{21} = -u_{12}, \tag{30}$$

$$u_{22} = \sin^2\theta e^{-i\lambda_1 \int_0^t J(t')dt'} + \cos^2\theta e^{-i\lambda_2 \int_0^t J(t')dt'}, \tag{31}$$

$$u_{33} = u_{44} = e^{2i\cos\phi_p \int_0^t J(t')dt'}, \tag{32}$$

where

$$\sin\theta = \sqrt{\frac{\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)} - \left(2\cos\phi_p + \frac{2}{\lambda}\right)}{2\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)}}}, \tag{33}$$

$$\cos\theta = \sqrt{\frac{\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)} + \left(2\cos\phi_p + \frac{2}{\lambda}\right)}{2\sqrt{\delta_p^2 + \left(2\cos\phi_p + \frac{2}{\lambda}\right)}}}. \tag{34}$$

III. SPIN CORRELATION FUNCTIONS AND ENTANGLEMENT EVALUATION

In this section, we evaluate the different magnetization and spin-spin correlation functions of the XY model, then we

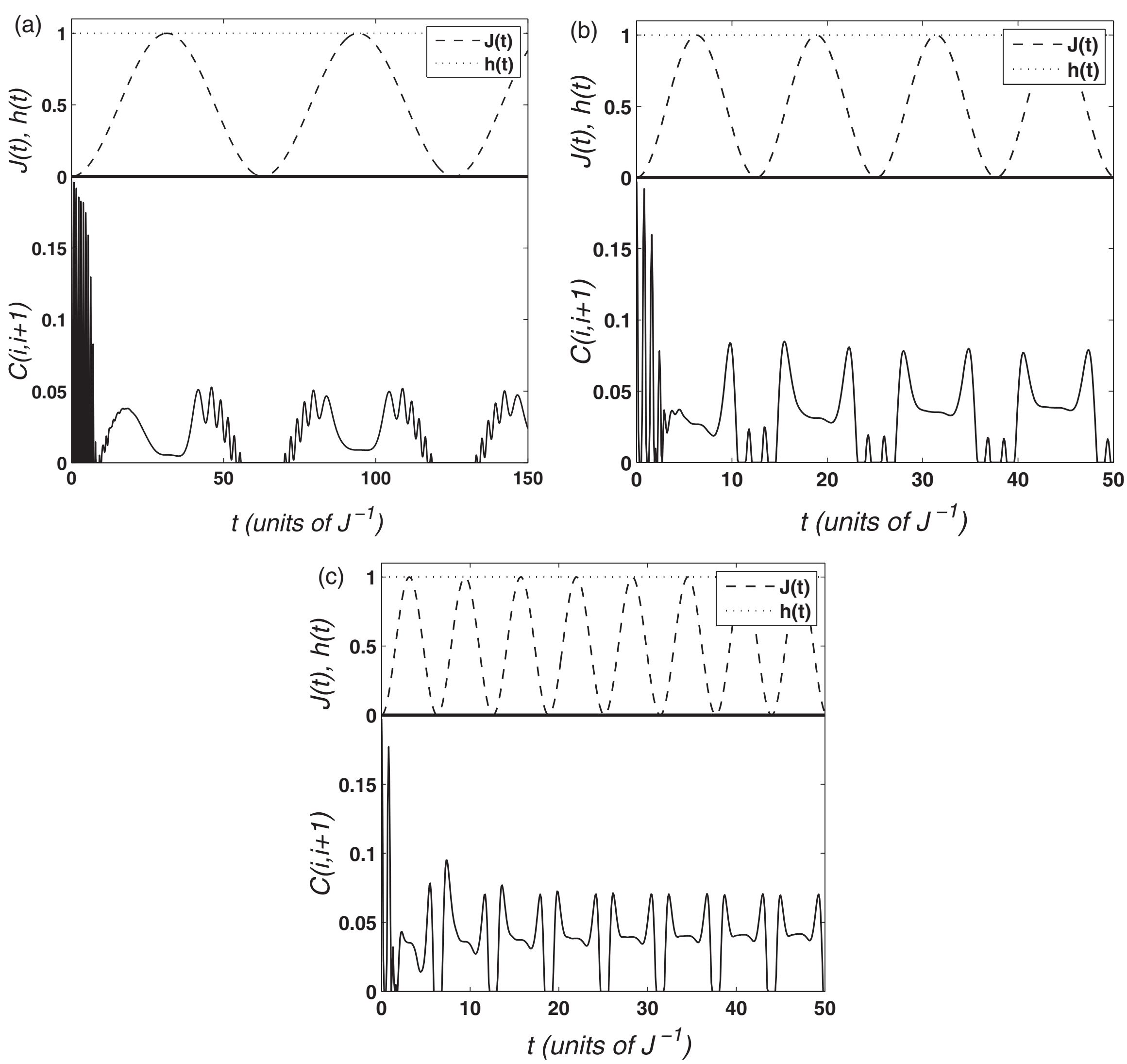

FIG. 3. Dynamics of nearest-neighbor concurrence with $\gamma = 1$ for $J_{\cos}$ where $J_0 = 0.5$, $h = 1$ at $kT = 0$ and (a) $K = 0.1$; (b) $K = 0.5$; and (c) $K = 1$.

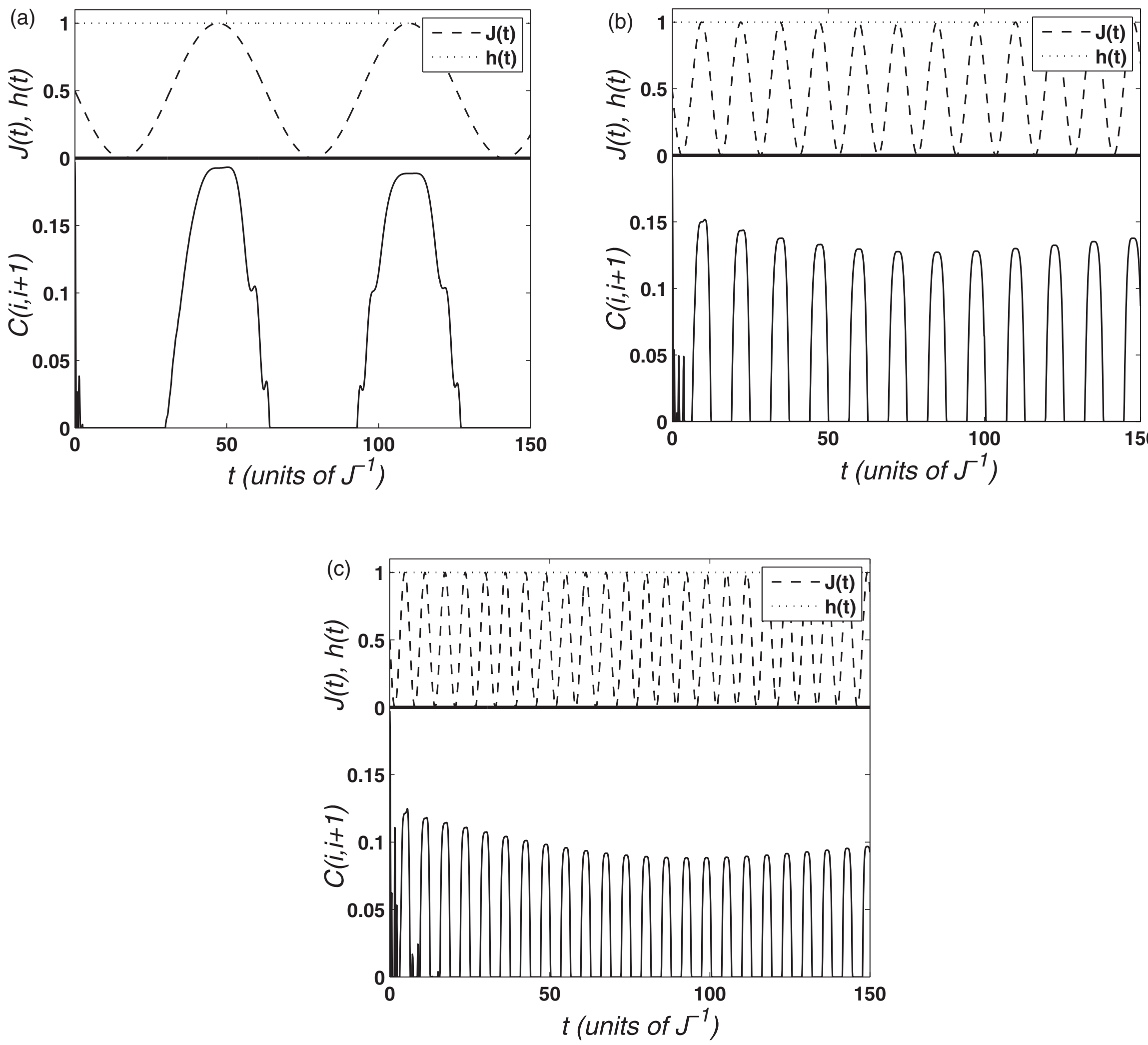

FIG. 4. Dynamics of nearest-neighbor concurrence with $\gamma = 1$ for $J_{\sin}$ with $J_0 = 0.5$, $h = 1$ at $kT = 0$ and (a) $K = 0.1$; (b) $K = 0.5$; and (c) $K = 1$.

evaluate the entanglement in the system. The magnetization in the z direction is defined as

$$M = \frac{1}{N}\sum_{j=1}^{N}\left(S_j^z\right) = \frac{1}{N}\sum_{p=1}^{1/N} M_p, \tag{35}$$

where $M_p = c_p^\dagger c_p + c_{-p}^\dagger c_{-p} - 1$. In terms of the density matrix, it is given by

$$\langle M_z\rangle = \frac{\mathrm{Tr}[M\rho(t)]}{\mathrm{Tr}[\rho(t)]} = \frac{1}{N}\sum_{p=1}^{1/N}\frac{\mathrm{Tr}[M_p\rho_p(t)]}{\mathrm{Tr}[\rho_p(t)]}. \tag{36}$$

The spin-correlation functions are defined by

$$S_{l,m}^x = \left\langle S_l^x S_m^x\right\rangle,\quad S_{l,m}^y = \left\langle S_l^y S_m^y\right\rangle,\quad S_{l,m}^z = \left\langle S_l^z S_m^z\right\rangle, \tag{37}$$

which can be written in terms of the fermionic operators as follows [52]:

$$S_{l,m}^x = \tfrac{1}{4}\left\langle B_l A_{l+1} B_{l+1}\cdots A_{m-1} B_{m-1} A_m\right\rangle, \tag{38}$$

$$S_{l,m}^y = \frac{(-1)^{l-m}}{4}\left\langle A_l B_{l+1} A_{l+1}\cdots B_{m-1} A_{m-1} B_m\right\rangle, \tag{39}$$

$$S_{l,m}^z = \tfrac{1}{4}\left\langle A_l B_l A_m B_m\right\rangle, \tag{40}$$

where

$$A_i = b_i^\dagger + b_i,\quad B_i = b_i^\dagger - b_i. \tag{41}$$

Using the Wick Theorem [55], the expressions (38)–(40) can be evaluated in terms of the Pfaffians as defined in Ref. [51]. For two neighbor spins l and $l+1$,

$$S_{l,l+1}^x = \tfrac{1}{4}F_{l,l+1}, \tag{42}$$

$$S_{l,l+1}^y = -\tfrac{1}{4}P_{l,l+1}, \tag{43}$$

and

$$S_{l,l+1}^z = \tfrac{1}{4}(P_{l,l}P_{l+1,l+1} - Q_{l,l+1}G_{l,l+1} + P_{l,l+1}F_{l,l+1}), \tag{44}$$

where

$$F_{l,m} = \langle B_l A_m\rangle,\quad P_{l,m} = \langle A_l B_m\rangle,\quad Q_{l,m} = \langle A_l A_m\rangle,\quad G_{l,m} = \langle B_l B_m\rangle. \tag{45}$$

To evaluate the entanglement between two quantum systems in the chain, we use the concurrence which has been

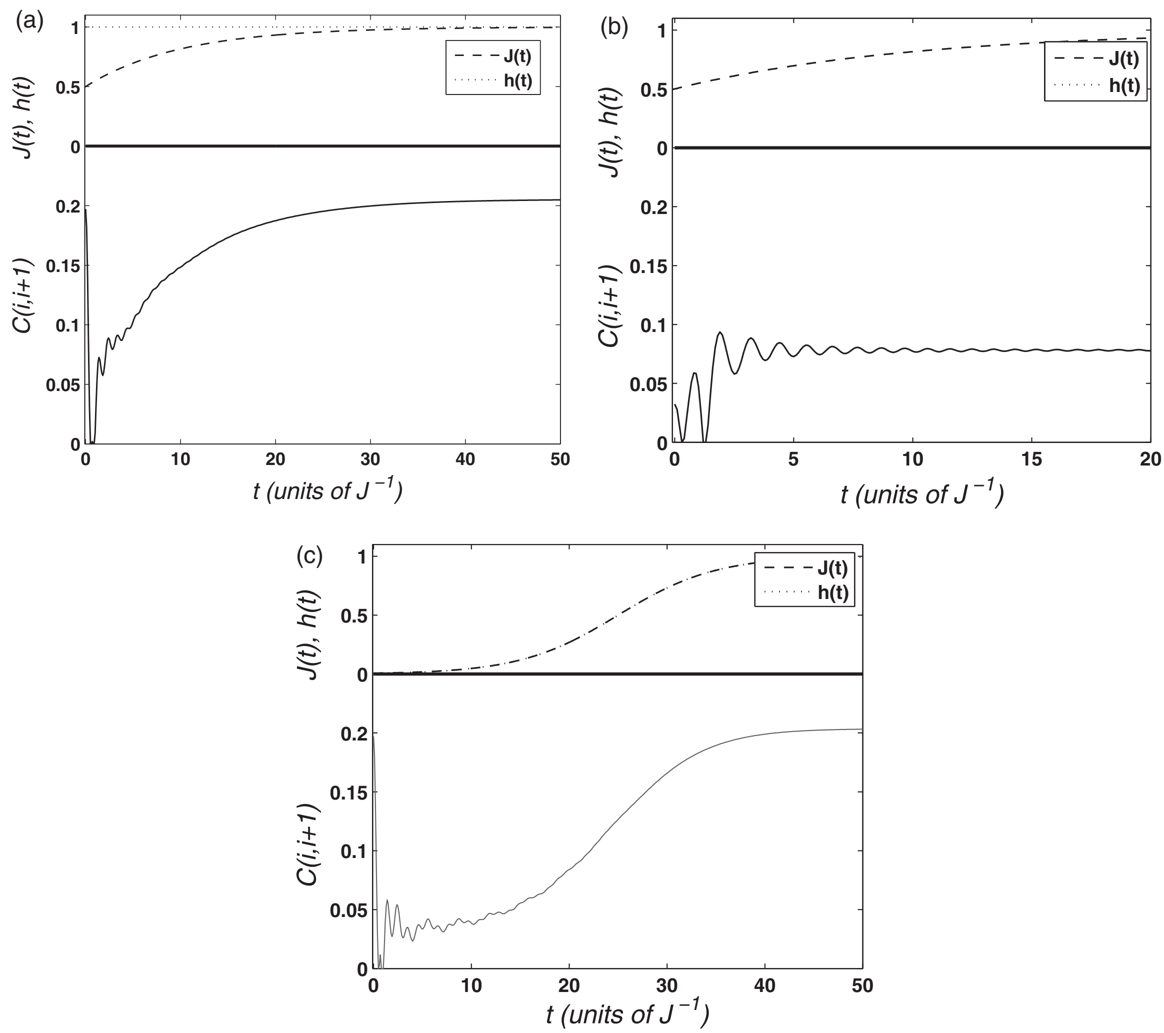

FIG. 5. Dynamics of nearest-neighbor concurrence with $\gamma = 1$ at $kT = 0$, $J_0 = 0.5$, $J_1 = 1$, $K = 0.1$ and (a) $h(t) = 1$; (b) $h(t) = J(t) = J_{\exp}$; and (c) $h(t) = J(t) = J_{\tanh}$.

shown to be a measure of entanglement [56]. The concurrence $C(t)$ is defined as

$$C(\rho) = \max(0, \lambda_a - \lambda_b - \lambda_c - \lambda_d), \tag{46}$$

where the λ_i's are the positive square root of the eigenvalues, in a descending order, of the matrix R defined by

$$R = \sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}, \tag{47}$$

and $\tilde{\rho}$ is the spin-flipped density matrix given by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y). \tag{48}$$

Knowing that ρ is symmetrical and real due to the symmetries of the Hamiltonian and particularly the global phase-flip symmetry, there will be only six nonzero distinguished matrix elements of ρ, which take the form [57]

$$\rho = \begin{pmatrix} \rho_{1,1} & 0 & 0 & \rho_{1,4} \\ 0 & \rho_{2,2} & \rho_{2,3} & 0 \\ 0 & \rho_{2,3} & \rho_{3,3} & 0 \\ \rho_{1,4} & 0 & 0 & \rho_{4,4} \end{pmatrix}. \tag{49}$$

Hence, the roots of the matrix R come out to be $\lambda_a = \sqrt{\rho_{1,1}\rho_{4,4}} + |\rho_{1,4}|$, $\lambda_b = \sqrt{\rho_{2,2}\rho_{3,3}} + |\rho_{2,3}|$, $\lambda_c = |\sqrt{\rho_{1,1}\rho_{4,4}} - |\rho_{1,4}||$, and $\lambda_d = |\sqrt{\rho_{2,2}\rho_{3,3}} - |\rho_{2,3}||$.

The nonzero matrix elements of ρ can be obtained, in the same way as in Ref. [51], in terms of the magnetization equation (36) and the spin-correlation functions (38)–(40).

IV. RESULTS AND DISCUSSION

A. Constant magnetic field

We start with studying the dynamics of the nearest-neighbor concurrence $C(i,i+1)$ for the completely anisotropic system $\gamma = 1$, when the coupling parameter is $J_{\exp}$ as well as $J_{\tanh}$ and the magnetic field is a constant using the numerical solution. In Fig. 1, we study the dynamics of the concurrence with the parameters $J_0 = 0.5$, $J_1 = 2$, $h = 1$, and different values of the transition constant $K = 0.1$ and 10. We note that the asymptotic value of the concurrence depends on K in addition to the coupling parameter and magnetic field. The larger the transition constant is, the lower the asymptotic value of the entanglement and the more rapid decay is. This result demonstrates the nonergodic behavior of the system, where

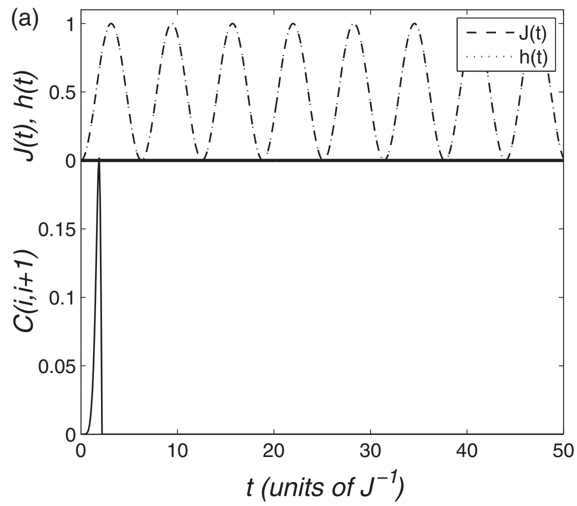

FIG. 6. Dynamics of nearest-neighbor concurrence with $\gamma = 1$ at $kT = 0$ with $J_0 = h_0 = 0.5$, $K = 1$ for (a) $J_{\cos}$ and $h_{\cos}$ and (b) $J_{\sin}$ and $h_{\sin}$.

the asymptotic value of the entanglement is different from the one obtained under constant coupling J_1.

In Fig. 2, we study the effect of the system size N on the dynamics of the concurrence. We select the parameters $J_0 = 0.5$, $J_1 = 2$, $h = 1$, and $K = 1000$. We note that, for all values of N, the concurrence reaches an approximately constant value but then starts oscillating after some critical time t_c that increases as N increases, which means that the oscillation will disappear as we approach an infinite one-dimensional system. Such oscillations are caused by the spin-wave packet propagation [48]. We next study the dynamics of the nearest-neighbor concurrence when the coupling parameter is $J_{\cos}$ with different values of K, i.e., different frequencies, which are shown in Fig. 3. We first note that $C(i,i+1)$ shows a periodic behavior with the same period of $J(t)$. It has been shown in a previous work [51] that, for the considered system at zero temperature, the concurrence depends only on the ratio J/h. When $J \approx h$, the concurrence has a maximum value. While when $J \gg h$ or $J \ll h$, the concurrence vanishes. In Fig. 3, one can see that when $J = J_{\max}$, $C(i,i+1)$ decreases because large values of J destroy the entanglement, while $C(i,i+1)$ reaches a maximum value when $J = J_0 = 0.5$. As $J(t)$ vanishes, $C(i,i+1)$ decreases because of the magnetic field domination. In Fig. 4, we study the dynamics of nearest-neighbor concurrence when $J = J_{\sin}$. As can be seen, $C(i,i+1)$ shows a periodic behavior with the same period as $J(t)$. We note that we get larger values of $C(i,i+1)$ compared to the previous case $J = J_{\cos}$. This indicates the importance of an initial concurrence to maintain and yield high concurrence as time evolves. Comparing our results with the previous results of time-dependent magnetic field [48], we note that the behavior of $C(i,i+1)$ when $J = J_{\cos}$ is similar to its behavior when $h = h_{\sin}$, where $h_{\sin} = h_0(1 - \sin(Kt))$, and vice versa.

B. Time-dependent magnetic field

In this section, we use the exact solution to study the concurrence for four forms of coupling parameter $J_{\exp}$, $J_{\tanh}$, $J_{\cos}$, and $J_{\sin}$ when $J(t) = \lambda h(t)$, where λ is a constant. We have compared the exact solution results with the numerical ones and they have shown coincidence. The dynamics of

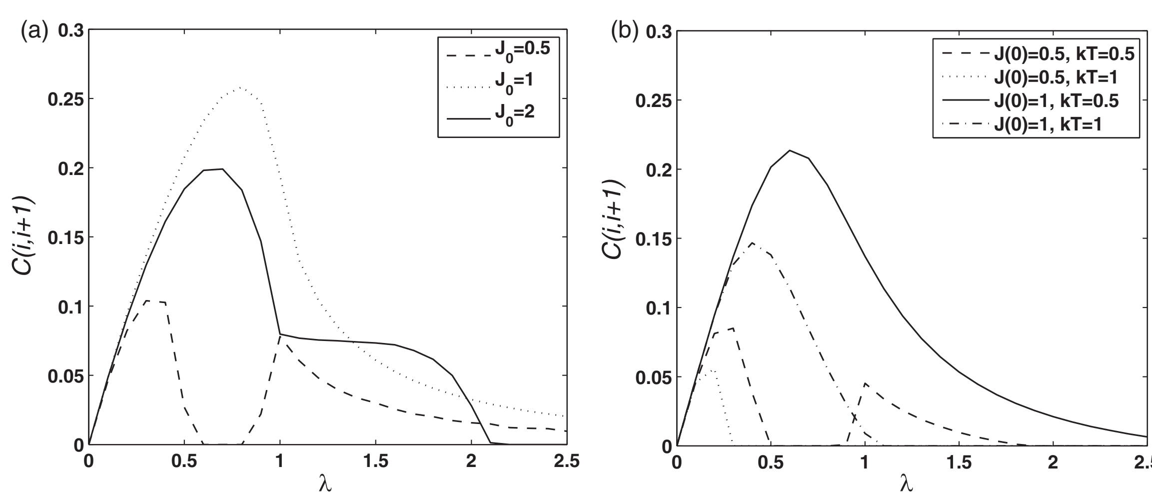

FIG. 7. The behavior asymptotic value of $C(i,i+1)$ as a function of λ with $\gamma = 1$ at (a) $kT = 0$ and (b) $kT = 0.5,1$.

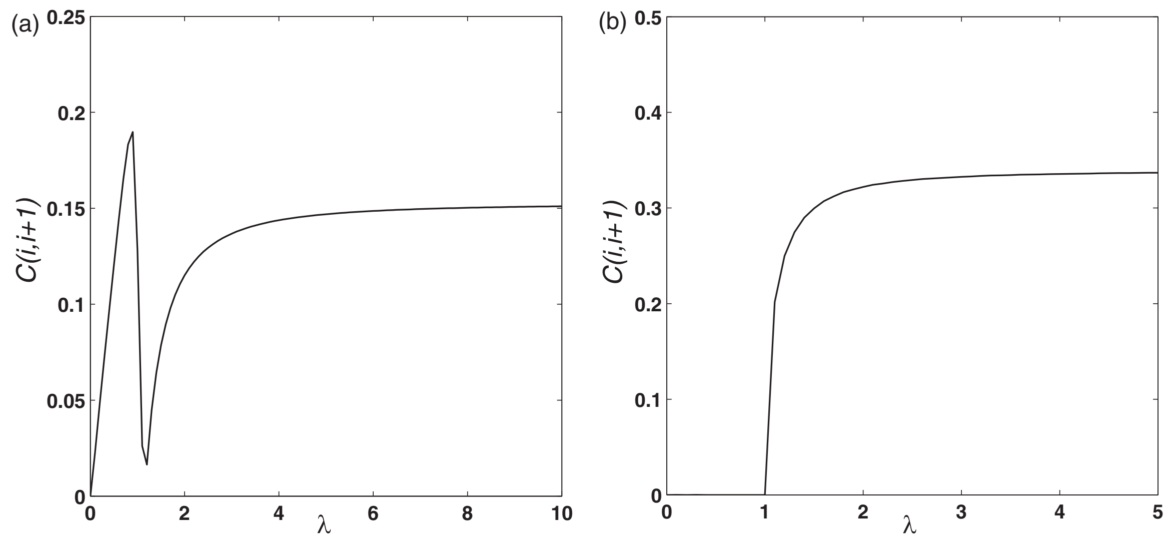

FIG. 8. The behavior asymptotic value of $C(i,i+1)$ as a function of λ at $kT = 0$ with (a) $\gamma = 0.5$ and (b) $\gamma = 0$.

$C(i,i+1)$ for $h(t) = 1$ and $J = J_{\text{exp}}$, $J_0 = 0.5$, $J_1 = 1$ with $K = 0.1$ is explored in Fig. 5(a). Comparing with Fig. 5(b), which shows the dynamics of $C(i,i+1)$ for $h(t) = J(t) = J_{\text{exp}}$, $J_0 = 0.5$, $J_1 = 1$, and $K = 0.1$, as one can see, the time-dependent magnetic field caused the asymptotic value of $C(i,i+1)$ to decrease. A similar behavior occurs when $h(t) = J(t) = J_{\text{tanh}}$, $J_0 = 0.5$, $J_1 = 1$ with $K = 0.1$ as exploited in Fig. 5(c). Figures 6(a) and 6(b) show the dynamics of $C(i,i+1)$ when $h(t) = J(t) = J_{\cos}$ and $h(t) = J(t) = J_{\sin}$, respectively, where $J_0 = 0.5$ and $K = 1$. As can be noticed, the concurrence in this case does not show a periodic behavior as it did when $h(t) = 1$ in Figs. 3 and 4.

In Fig. 7(a), we study the behavior of the asymptotic value of $C(i,i+1)$ as a function of λ at different values of the parameters J_0, J_1, and K where $J(t) = \lambda h(t)$. Interestingly, the asymptotic value of $C(i,i+1)$ depends only on the initial conditions, not on the form or behavior of $J(t)$ at $t > 0$. This result demonstrates the sensitivity of the concurrence evolution to its initial value. Testing the concurrence at nonzero temperatures demonstrates that it maintains the same profile, but with reduced value with increasing temperature as can be concluded from Fig. 7(b). Also, the critical value of λ at which the concurrence vanishes decreases with increasing temperature as can be observed, which is expected as thermal fluctuations destroy the entanglement. Finally, in Fig. 8, we study the partially anisotropic system $\gamma = 0.5$ and the isotropic system $\gamma = 0$ with $J(0) = 1$. We note that the behavior of $C(i,i+1)$ in this case is similar to the case of constant coupling parameter studied previously [51]. We also note that the behavior depends only on the initial coupling $J(0)$ and not on the form of $J(t)$ where different forms have been tested. It is interesting to notice that the results of Figs. 6, 7, and 8 confirm one of the main results of the previous works [47,48,51], namely, that the dynamic behavior of the spin system, including entanglement, depends only on the parameter $\lambda = J/h$, not the individual values of h and J for any degree of anisotropy of the system. In these previous works, both the coupling and magnetic field were considered time independent, while in this paper we have assumed $J(t) = \lambda h(t)$, where $h(t)$ can take any time-dependent form. This explains why the asymptotic value of the concurrence depends only on the initial value of the parameters regardless of their function form. Furthermore, in the previous works, it was demonstrated that, for finite temperatures, the concurrence turns out to be dependent not only on λ but on the individual values of h and J, while according to Fig. 7(b), even at finite temperatures, the concurrence still depends only on λ where $J(t) = \lambda J(t)$ for any form of $h(t)$.

V. CONCLUSIONS AND FUTURE DIRECTIONS

We have studied the dynamics of entanglement in a one-dimensional XY spin chain coupled through a time-dependent nearest-neighbor coupling and in the presence of a time-dependent magnetic field at zero and finite temperatures. We presented a numerical solution for the system for general forms of $J(t)$ and $h(t)$ and an exact solution for proportional $J(t)$ and $h(t)$. For an exponentially increasing $J(t)$, we found that the asymptotic value of the concurrence depends on the exponent transition constant value, which confirms the nonergodic behavior of the system. For a periodic $J(t)$, we found that the concurrence shows a periodic behavior with the same period as $J(t)$. On the other hand, for both periodic coupling and magnetic field with same period, the concurrence loses its periodic behavior. When $J(t) = \lambda h(t)$, where λ is a constant, we found that the the dynamical behavior and asymptotic value of the concurrence depends only on the initial conditions, regardless of the form of the coupling parameter (and the magnetic field). This confirms the results of previous works that the dynamical behavior of the system depends only on the parameter $\lambda = J/h$ at zero temperature for any degree of anisotropy. Furthermore, in this paper, we have demonstrated that this character is still valid even in the case of proportional time-dependent coupling and magnetic field at zero and finite temperatures. This result also explains the nonoscillatory behavior of concurrence although both the coupling and the field are oscillating. In the future, we would like to study the effect of an impurity spin on the entanglement along the driven one-dimensional spin chain. It will be also interesting to study the decoherence of a spin pair (quantum gate) as a result of coupling to a driven one-dimensional spin chain acting as its environment.

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