Dimensional scaling as a symmetry operation

S. Kais and D. R. Herschbach
Department of Chemistry, Harvard University, Cambridge, Massachusetts 02138

R. D. Levine
The Fritz Haber Research Center for Molecular Dynamics, The Hebrew University,
Jerusalem 91904, Israel

(Received 3 July 1989; accepted 7 September 1989)

The scaling of the Schrödinger equation with spatial dimension $D$ is studied by an algebraic approach. For any spherically symmetric potential, the Hamiltonian is invariant under such scaling to order $1/D^2$. For the special family of potentials that are homogeneous functions of the radial coordinate, the scaling invariance is exact to all orders in $1/D$. Explicit algebraic expressions are derived for the operators which shift $D$ up or down. These ladder operators form an SU(1,1) algebra. The spectrum generating algebra to order $1/D^2$ corresponds to harmonic motion. In the $D \to \infty$ limit the ladder operators commute and yield a classical-like continuous energy spectrum. The relation of supersymmetry and $D$ scaling is also illustrated by deriving an analytic solution for the Hooke’s law model of a two-electron atom, subject to a constraint linking the harmonic frequency to the nuclear charge and the dimension.

I. INTRODUCTION

For problems which lack a natural perturbation expansion parameter, it is sometimes advantageous to regard the number of space dimensions as a variable. The hydrogenic atom, the harmonic oscillator, and the quark charmonium model are examples where the physical constants in the Hamiltonian serve only to set the distance and energy scales, so the reduced Hamiltonian has no remaining free parameters. This situation, which obtains for any potential that is a homogeneous function of a single radial coordinate, has fostered interest in $1/D$ perturbation expansions. Prototype applications include many central force problems, quantum spin models, critical phenomena, and electronic structure of atoms and molecules.

This paper pursues an improved understanding of dimensional scaling by considering it as a symmetry operation. Thus, we examine Hamiltonians that are invariant under a scaling transformation. Even for problems involving a single radial coordinate, we find that only for a subclass of potentials (including, in particular, the hydrogenic atom and the harmonic oscillator) can this invariance be exact. However, it is always exact through order $1/D$; to this order, all potentials admit of dimensional scaling.

Our treatment is Lie algebraic, but differs in an essential way from a previous algebraic approach by Mlodinow and Papanicolaou. In their work, the $1/D$ expansion was considered for the spectrum generating algebra. Here we use generators that shift $D$. This is an extension of work by Wulfman and Levine which introduced mass scaling in order to study the invariance properties of the bound state spectrum under isotopic substitution.

In Sec. II we study the scaling of the dimension and other variables and determine the explicit form of the transformation that leaves the Hamiltonian invariant (at least to order $1/D$). In Sec. III the corresponding Lie generators are shown to close an SU(1,1) algebra. We consider in Sec. IV the scaling of the spectrum with dimension and in Sec. V illustrate the relation of $D$ scaling to supersymmetry by deriving an analytic solution for the Hooke’s law model for a two-electron atom.

II. DIMENSIONAL SCALING

We begin with some preliminary comments about scale transformations and the implied invariance properties of the Schrödinger equation and then proceed to examine the symmetry transformation of the radial Hamiltonian.

A. Scale transformation

A transformation $T$ is a symmetry of the Hamiltonian $H$ when $THT^{-1} = H$. It follows that if $\Psi$ is an eigenfunction of $H$ with the eigenvalue $E$ then $T\Psi$ is also an eigenfunction with the same eigenvalue. Thus, $T$ is an ordinary symmetry and is a special case of the more general dynamical symmetry (also used below) which converts solutions into other, not necessarily degenerate solutions. In terms of the generator $R$ of the transformation, $T = e^R$, an ordinary symmetry implies that $R$ and $H$ commute,

$$[R,H] = 0. \quad (2.1)$$

We shall be particularly interested in such generators $R$ that are first order differential operators. Since $R = x \partial / \partial x$ is the scale or dilation generator for the variable $x$, the corresponding transformation is

$$e^{x \partial / \partial x} f(x) = f(e^x x), \quad (2.2)$$

where $f(x)$ is any analytic function of $x$. This may be verified by expanding $e^R$ as a power series and rearranging it to the Taylor expansion of $f(e^x x)$. For such generators $R$, the symmetry condition $(2.1)$ can be written as $RH = 0$.

Equation $(2.1)$ needs to be modified when $R$ is a generator of a generalized symmetry. The generator $R'$ which leaves the eigenfunctions $\Psi$ of $H$ invariant and scales the eigenvalues by $e$ satisfies

$$R'H = eH \quad (2.3)$$
as can be verified by operating on $\Psi$ with both sides, 
\((R'|H)\Psi = \epsilon E \Psi\). Note that if \(R'\) scales $\epsilon$ as well such that 
\(R'e = \epsilon e\) then Eq. (2.3) implies 
\[ R'(H/\epsilon) = 0 \] 
so that \(R'\) is a generator of an ordinary symmetry transformation for the reduced Hamiltonian $H/\epsilon$.

**B. The symmetry of the radial Hamiltonian**

The radial Schrödinger equation in $D$ space dimensions for an arbitrary spherically symmetric potential is 
\[ \left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} - \frac{\Lambda(\Lambda + 1)}{r^2} + V(r) \right\} \Psi = E \Psi, \] 
where $\Lambda = l + \frac{1}{2}(D - 3)$ and $l$ is the orbital angular momentum. In certain applications it proves convenient to introduce a shift in the variable $\Lambda$ and work with $\Lambda - \Lambda_0$, where the shift $\Lambda_0$ can depend on energy (and other variables). We shall not examine this variant here. Note that we follow the customary procedure for central force problems in which the square root of the Jacobian is tucked into the probability amplitude. Accordingly, the transformed amplitude $\Psi$ has unit Jacobian and the corresponding Laplacian is much simplified. The radial part of the original Jacobian, which goes as $r^{D-1}$, gives an additional contribution to the $r^{-2}$ term in the transformed Laplacian. This combines with the eigenvalue of the square of the angular momentum, $(l + D - 2)$, to yield the factor $\Lambda(\Lambda + 1)$ in the centrifugal term.

We consider first the family of potentials $V(r)$ for which the effective potential in the radial equation can be recast in terms of range and strength parameters, 
\[ \left(\frac{\hbar^2}{2m}\right) \Lambda(\Lambda + 1)/r^2 + V(r) \equiv \epsilon U(\alpha r). \] 
(2.6)

For the particular case of attractive inverse power potentials, 
\[ V(r) = -C_n r^{-n}, \] 
(2.5c)

with $C_n > 0$, this holds with 
\[ \alpha = \left[ \frac{\hbar^2}{2m} \Lambda(\Lambda + 1)/C_n \right]^{1/(n-2)} \] 
(2.7a)

and 
\[ \epsilon = C_n \alpha^n = \left( \frac{\hbar^2}{2m} \Lambda(\Lambda + 1) \right)^{n/(n-2)} C_n^{-2/(n-2)}. \] 
(2.7b)

The radial Hamiltonian thus contains three parameters $(m, \epsilon, \alpha)$ and one variable. The generator that scales all four has the general form
\[ R = c_1 \frac{\partial}{\partial r} + c_2 \frac{\partial}{\partial m} + c_3 \frac{\partial}{\partial \epsilon} + c_4 \frac{\partial}{\partial \alpha}. \] 
(2.8)

If $R$ is to be a symmetry, $R'H = 0$ and substituting the form of the radial Hamiltonian results in four equations for the four coefficients $c_i$, $i = 1, \cdots, 4$. To evaluate the action of $R$ on $H$ note that 
\[ \begin{bmatrix} \frac{\partial^2}{\partial r^2} & \frac{\partial}{\partial r} \\ \frac{\partial}{\partial r} & \frac{\partial^2}{\partial r^2} \end{bmatrix} = -2 \frac{\partial^2}{\partial r^2} \] 
and that 
\[ \frac{\partial U(\alpha r)}{\partial \alpha} = \frac{\partial U(\alpha r)}{\partial r}. \] 
Hence, Eq. (2.1) implies 
\[ c_2 = -2c_1, \quad c_3 = 0, \quad c_4 = -c_1, \] 
and the symmetry generator is 
\[ R = r \frac{\partial}{\partial r} - 2m \frac{\partial}{\partial m} - \alpha \frac{\partial}{\partial \alpha}. \] 
(2.9)

The symmetry $R$ leaves invariant not only the Hamiltonian but also the canonical distance and mass variables 
\[ \rho = ar \quad \text{and} \quad \lambda = \alpha \left(\frac{\hbar^2}{2m}\right)^{1/2} \] 
since $R\rho = 0$ and $R\lambda = 0$. However, $R$ is only a harbinger of the symmetry known as the "law of corresponding states" to which we now turn. Our purpose is to determine the generator $R'$ that leaves the reduced Hamiltonian $H/\epsilon$ invariant, 
\[ R'(H/\epsilon) = 0. \] 
Since $H = -m\dot{\theta}^2 H/\partial m + \epsilon \dot{\theta} H/\partial \epsilon$, it follows from Eq. (2.3) that 
\[ R' = R + \epsilon \left( \frac{\partial}{\partial \epsilon} - m \frac{\partial}{\partial m} \right). \] 
(2.11)

This form for $R'$ leaves invariant the parameter 
\[ B = 2m e^{2}/\hbar^2 \] 
(2.12)

since $R'B = 0$. This quantity, known as the "well capacity parameter," determines the number of bound states. Note that in the present context, however, the angular momentum $l$ need not be zero and $D$ is not necessarily equal to 3. Indeed, it follows from Eq. (2.6) that $\epsilon/(\alpha r)^2$ is the effective centrifugal term so that 
\[ B = \Lambda(\Lambda + 1) \] 
(2.13)

and $B \to \infty$ as $D \to \infty$.

In terms of the canonical variables $\rho$ and $B$ which are invariant under $R'$ the radial Schrödinger equation (2.5) now reads 
\[ \left[ -\frac{1}{B} \frac{\partial^2}{\partial \rho^2} + U(\rho) \right] \Psi(\rho) = (E/\epsilon) \Psi(\rho). \] 
(2.14)

Here $U(\rho)$ is the scaled effective potential introduced in Eq. (2.6), which depends only on $\rho$ and has no explicit $D$ dependence. For attractive inverse power potentials, where $V(r) \sim -r^{-n}$, the scaled effective potential is simply 
\[ U(\rho) = \frac{1}{\rho^2} - \frac{1}{\rho^n}, \] 
(2.15)

Of course, Eq. (2.14) will not hold for potentials that do not satisfy Eq. (2.6), which requires that $V(r)$ be a homogeneous function of the radial coordinate. In particular, many potentials will satisfy (2.6) for $l = 0$ and $D = 3$ but not for a nonzero $l$ or $D > 3$. In the general case, $U(\rho)$ in Eqs. (2.6) and (2.14) must be replaced by $U + \Delta U$, where the incremental potential $\Delta U = \epsilon^{-1} \Delta V$ arising from nonhomogeneous terms and the corresponding reduced Hamiltonian are not left invariant by $R'$. However, the symmetry breaking is proportional to $\epsilon^{-1}$ to leading order, 
\[ R'(H/\epsilon) = 0 + O(\epsilon^{-1} \Delta V). \] 
(2.16)

As seen in Eq. (2.7), in the large $D$ limit $\epsilon^{-1} \to 0$, so the symmetry breaking then becomes negligible.

For $l = 0$ in three dimensions $\Lambda = 0$ and many familiar potentials will satisfy Eq. (2.6) exactly, i.e., with $\Delta V = 0$ (and appropriate $\alpha$ and $\epsilon$ scaling parameters not involving $\Lambda$). This is the familiar version of the law of corresponding states. For $\Lambda > 0$, the increment $\Delta V$ need not be zero, but often simplifying approximations are suitable, depending on.
which range of \( \rho \) values is most important. Such approximations include: (1) taking \( \Delta V \) to be the short range part of \( V(r) \), as in the Landau–Lifshitz approximation for the total scattering cross section; (2) expanding the potential in a Taylor series about the minimum and taking for \( \Delta V \) all terms beyond the quadratic; (3) developing power series in \( 1/D \) by expanding \( 1/r^2 \) about the minimum of \( V(r) \), as in the Peierls approximation.

III. THE DIMENSIONAL SCALING GROUP

We now proceed to identify SU(1,1) as the group which acts to change the dimension in a continuous manner. This is equivalent to applying the same group to continuous changes of the potential well depth, and is to be distinguished from the so called “potential group” which acts so as to change the well depth by integer amounts.

According to Eqs. (2.10) and (2.12), the energy scale parameter \( \epsilon = a^2 B \). If for simplicity we put \( B = b^2 \), the generator which leaves \( \epsilon \) invariant is of the form

\[
K_1 = -i \left( b \frac{\partial}{\partial b} - a \frac{\partial}{\partial a} \right) \tag{3.1}
\]

since \( K_1 \epsilon = 0 \). With the two additional generators

\[
K_{3,2} = -i \left[ (b^2 \pm 1) \frac{\partial}{\partial b} \pm (a^2 \pm 1) \frac{\partial}{\partial a} \right] \tag{3.2}
\]

where \( + \leftrightarrow 3 \) and \( - \leftrightarrow 2 \), we have the structure for the SU(1,1) group,

\[
\{ K_1, K_2 \} = -i K_3; \{ K_2, K_3 \} = -i K_1; \{ K_3, K_1 \} = i K_2. \tag{3.3}
\]

That the action of the dimensional scaling group is to scale \( B \) in the reduced Hamiltonian \( H/\epsilon \) follows from Eq. (3.2) and from

\[
T_i B = e^{2i B}, \tag{3.4}
\]

where \( T_i = \exp(ik K_i) \) for any real group parameter \( k \). It follows that

\[
T_i (H/\epsilon) \rightarrow H'/\epsilon, \tag{3.5}
\]

where the value of the \( B \) parameter for \( H'/\epsilon \) is given by Eq. (3.5).

IV. THE SPECTRUM GENERATING GROUP

Corresponding to the radial equation (2.14) we can write the reduced Hamiltonian

\[
H/\epsilon = -\frac{1}{B} \frac{d^2}{dr^2} + U(\rho). \tag{4.1}
\]

Such a Hamiltonian can always be factored as\(^{13}\)

\[
H/\epsilon = A^+ A^- + [A^-, A^+, \tag{4.2}
\]

where

\[
A^\pm = \mp B - \frac{1}{2} \frac{dF(\rho)}{d\rho} \tag{4.3}
\]

and

\[
[A^-, A^+] = 2B - \frac{1}{2} \frac{dF(\rho)}{d\rho} \tag{4.4}
\]

with \( F(\rho) \) defined by

\[
U(\rho) = F^2(\rho) + B - \frac{1}{2} \frac{dF(\rho)}{d\rho}. \tag{4.5}
\]

A. The large \( D \) limit

In the \( D \rightarrow \infty \) limit, \( B \rightarrow D^2/4 \) so the solution of the Riccati equation (4.5) is trivial as the last term can be neglected. Also, in the \( D \rightarrow \infty \) limit, the ladder operators \( A^- \) and \( A^- \) will commute, and the spectrum of \( H \) degenerates to one value (since \( H \) and \( A^+ \) commute), corresponding to the system being at the bottom of the effective potential, \( U(\rho) \). However, this localization does not violate the uncertainty principle.\(^{6}\) Scaling the radial coordinate by a dimension-dependent factor implies that the conjugate momentum is scaled inversely, so that the commutator and the uncertainty principle remain invariant.

In many cases of interest the factorization can be obtained explicitly to first order in \( 1/D \). Whenever the effective potential \( U(\rho) \) has a minimum, we can take the limit \( B \rightarrow \infty \) while keeping \( Ba^4 \) finite. This corresponds to keeping just the harmonic term in the Taylor expansion of \( U(\rho) \) about its minimum. From Eqs. (2.10) and (2.12), the limit corresponds to \( \epsilon \rightarrow \infty, a \rightarrow 0 \) such that \( ea^2 \) is finite, in terms of the depth and range parameters for \( U(\rho) \) introduced in Eq. (2.6). For this harmonic limit the effective potential no longer has two scale parameters, but just one, namely \( ea^2 \). In the customary spectroscopic notation, the harmonic frequency is \( \omega_0 = (4B)^{1/2} a^2 \) and the quartic anharmonicity is \( \omega_0 x_0 = a^2 \). The relevant reduced Hamiltonian is not \( H/\epsilon \) but \( H/\omega_0 \), since the scaling of the harmonic frequency is governed by \( ea^2 \).

B. Coulombic \( D \)-shift operators

To illustrate \( D \) scaling via the factorization procedure and the harmonic limit, we consider the hydrogenic atom, corresponding to \( n = 1 \) and \( C_n = Ze^2 \) in Eq. (2.6); thus the scale parameters are \( \alpha = [2Z/\Lambda(\Lambda + 1)]/a_0 \) and \( \epsilon = [2Z^2/\Lambda(\Lambda + 1)](e^2/a_0) \), where \( a_0 = \hbar^2/me^2 \) is the Bohr radius. In view of the relation \( \Lambda = l + \frac{1}{2}(D - 3) \), the ladder operators\(^{14} \) that shift \( / \pm \) are equivalent to operators that shift \( D \) by \( \pm 2 \) units. These operators, which factorize Eq. (2.5), are given by

\[
A^\pm = (\hbar^2/2m)^{1/2} \left[ \frac{d}{dr} - \frac{(\Lambda + 1)}{r} + \frac{Z}{(\Lambda + 1)a_0} \right]. \tag{4.6}
\]

The factorized Hamiltonian can be written as

\[
H = A^+ A^- + \gamma_0, \tag{4.7}
\]

where \( \gamma_0 = -\frac{1}{4}(Z^2e^2/a_0)(\Lambda + 1)^{-2} \). The corresponding commutator is

\[
[A^-, A^+] = (\hbar/m)[(\Lambda + 1)/r^2] = H_{\Lambda + 1} - H_{\Lambda}, \tag{4.8}
\]

and the eigenvalue is

\[
E_\Lambda = \left[ Z/(k + \Lambda + 1)^2 \right](e^2/a_0), \tag{4.9}
\]

where \( k = 0,1,2, \cdots \) counts the number of radial nodes.

We wish to show that the commutator (4.8) becomes unity in the harmonic limit, \( \epsilon \rightarrow \infty, a \rightarrow 0 \). When applied to the eigenfunctions, the commutator gives \( E_{\Lambda + 1} - E_\Lambda \). The
eigenvalue difference is readily evaluated in terms of $\epsilon$ and $\alpha$, since
\[
\Lambda + 1 = \frac{1}{2}(1 \pm (1 + 4B)^{1/2})
\] (4.10)
and Eq. (2.12) yields
\[
E_{\Lambda} = -(2m/\hbar^2) \epsilon \{ (2k + 1) \alpha \\
\pm [\alpha^2 + (2m/\hbar^2) \epsilon]^{1/2} \}^{-2}
\] (4.11)
On expanding the eigenvalue difference in a Taylor series about $\alpha = 0$ and taking the limit $\epsilon \to \infty, \alpha \to 0$ with $\alpha^2$ held constant, we find the harmonic frequency $\omega_\alpha = (\hbar/2m)^{1/2} \epsilon^{1/2}$. Since the relevant reduced Hamiltonian is $H/\omega_\alpha$, the ladder operators are rescaled by $\omega_\alpha^{-1/2}$ such that the commutator $[A^-, A^+]$ is unity.

**C. The Dunham expansion**

Analogous procedures are applicable when the potential is not a homogeneous function of the radial coordinate. Here we note, as a practical matter, that the widely used Dunham approximation for the energy levels of a diatomic molecule can be readily converted to an $1/D$ expansion. In his treatment (for $D = 3$), the Hamiltonian of Eq. (2.5) is expanded in powers of $\xi = (r - r_i)/r_i$, and the energy levels evaluated in the form
\[
E = \Sigma_{m} Y_{nm}(v + \frac{1}{2})^n \{ \Lambda(\Lambda + 1) \}^m
\] (4.12)
with the vibrational quantum number. Expressions are available for the $Y_{nm}$ coefficients (with $n = 0$ to 6 and $m = 0$ to 6) in terms of the rotational constant $B_\ast = \hbar^2/(2mr_i^2)$, its ratio to the harmonic frequency, $B_\ast/\omega_\alpha$, and the potential parameters $a_k$ that appear in the Taylor expansion,
\[
V(r) = a_0 \xi^2 + (a_1 \xi + a_2 \xi^2 + \cdots)
\] (4.13)
To obtain the corresponding dimensional perturbation expansion, it is only necessary to substitute in Dunham's expressions the appropriate $a_k$ coefficients and to evaluate $B_\ast$ and $B_\ast/\omega_\alpha$ as functions of $\Lambda = l + 1/2(D - 3)$. For instance, for a Coulombic potential $a_k = (-1)^k(k + 1)$, the rotational constant is $B_\ast = 2\Lambda(\Lambda + 1) - 1$ and $B_\ast/\omega_\alpha = \Lambda(\Lambda + 1)^{-1/2}$, in reduced units. This offers an efficient procedure for obtaining $1/D$ expansions, since the Dunham coefficients have been evaluated for a host of potential functions, including the Morse potential, the Hulbert–Hirschfelder potential, and many others. Analytic recurrence relations have also been developed which enable the set of $Y_{nm}$ coefficients to be generated from the $Y_{nm}$ coefficients.

**V. D SCALING AND SUPERSYMMETRY**

A pair of one-dimensional potentials $V_{\pm}(r)$ is said to be supersymmetric partner potentials if they have the same energy levels, $E_{\pm}^n = E_{\mp}^{n+1}$, except for the ground state, which is taken as the energy zero, $E_{\pm}^0 = 0$. The existence of such partner potentials can be exploited to simplify evaluation of $1/D$ perturbation expansions and improve the accuracy. Here we illustrate a different way in which supersymmetry can be used to enhance $D$ scaling. We obtain an analytic solution for the Hooke's law model of a two-electron atom, subject to a constraint linking the harmonic frequency to the nuclear charge and the dimension.

**A. Factorization scheme**

The pair of Hamiltonians pertaining to supersymmetry is
\[
H_\pm = -d^2/dr^2 + V_\pm(r)
\] (5.1)
and the potentials are related by
\[
V_\pm(r) = W^2(r) \pm \frac{dW}{dr}
\] (5.2)
so
\[
V_\pm(r) = V_-(r) + 2 \frac{dW}{dr},
\] (5.3)
where $W(r)$ is called the superpotential. For the potential $V_-(r)$ the ground state wave function $\Phi_0^-$ is assumed to be known; also, an additive constant is included in the potential to make the ground state eigenvalue vanish, so $H_\pm \Phi_0^- = 0$ and $E_0^- = 0$. Then if the operators are defined by
\[
A^\pm = \pm \frac{d}{dr} + W(r),
\] (5.4)
the pair of supersymmetric Hamiltonians is factorized as $H_\pm = A^+ A^- \pm A^- A^+$ and $H_\mp = A^+ A^-$. This scheme corresponds to the factorization procedure (4.2)–(4.5) used for the spectrum generating group.

The pair of potentials $V_\pm(r)$ specified by Eq. (5.3) are said to be shape invariant if they are similar in form and differ only in the values of the parameters that appear in them; potentials with this property are analytically solvable. For the $D$-dimensional hydrogenic atom and harmonic oscillator, for instance, the $V_\pm$ potentials can readily be shown to be shape invariant.

**B. Hooke's law model for two-electron atom**

In this model, the electrons are attracted to the nucleus by an harmonic oscillator potential but repel each other by the usual Coulombic force. The corresponding Hamiltonian is
\[
H = - (\hbar^2/2m)(\nabla_1^2 + \nabla_2^2) + i\hbar \omega \kappa(r_1^2 + r_2^2) + e^2/r_{12},
\] (5.5)
where $m$ is the reduced mass of the electron and $\omega$ the angular frequency of the electron-nucleus oscillator. On choosing units such that $\hbar = m = e = 1$ and scaling the coordinates by $r \to Zr_1$, the Schrödinger equation becomes $H \Psi = e \Psi$, with
\[
H = -i(\nabla_1^2 + \nabla_2^2) + \beta^2 (r_1^2 + r_2^2) + \lambda / \rho_{12},
\] (5.6)
where $\beta^2 = \omega^2/Z^4, \lambda = 1/Z$, and $e = E/Z^2$. Perturbation treatments of the Hooke's law model take $\beta = 1$, but here we keep $\beta$ as a free parameter to be determined later. Transforming to centroid and relative position coordinates,
\[
R = 2^{-1/2}(r_2 + r_1) \quad \text{and} \quad \rho = 2^{-1/2}(r_2 - r_1)
\] (5.7)
separates Eq. (5.6) into a pair of equations,

\[ H_{R}^{\chi} = \epsilon_{R}^{\chi} \mathbf{\chi}(\mathbf{R}) \text{ and } H_{\rho}^{\psi} = \epsilon_{\rho}^{\psi} \mathbf{\psi}(\rho) \] 
\[ \text{with } H_{R} = -\frac{1}{2} \nabla_{R}^{2} + \frac{1}{2} \beta^{2} R^{2} \] 
\[ H_{\rho} = -\frac{1}{2} \nabla_{\rho}^{2} + \frac{1}{2} \beta^{2} \rho^{2} + \frac{\lambda}{\rho}, \] 
\[ \text{where } \lambda = 2^{-1/2} \lambda \text{ and } \] 
\[ \mathbf{\psi}(r_{1}, r_{2}) = \chi(R) \Psi(\rho) \text{ and } \epsilon = \epsilon_{R} + \epsilon_{\rho}. \] 

The generalization to D dimensions merely involves introducing D dimensional harmonic oscillators, with eigenvalues given by

\[ \epsilon_{n} = \beta(2n + l + \frac{1}{2}D), \quad \text{where } n = 0, 1, 2, \cdots. \] 

The solution of Eq. (5.10) is simplified by

\[ \Psi(\rho) = \rho^{-\frac{1}{2}(D - 1/2)} \Psi(\rho), \] 

which yields \( \mathbf{\hat{H}}_{\rho} \Psi = \epsilon \Psi \) with

\[ \mathbf{\hat{H}}_{\rho} = -\frac{1}{2} \frac{d^{2}}{d\rho^{2}} + \frac{\Lambda(\Lambda + 1)}{2\rho^{2}} + \frac{1}{2} \beta^{2} \rho^{2} + \frac{\lambda}{\rho}, \] 

where again \( \Lambda = l + \frac{1}{2}(D - 3) \). To obtain the ground state energy, we employ the operators of Eq. (5.4), with \( r \) replaced by \( \rho \), and take

\[ W(\rho) = \frac{a}{\rho} + b \rho - \frac{g}{1 + g\rho}, \] 

where \( a, b, \) and \( g \) are constants to be determined. Then

\[ V_{-}(\rho) = W^{2} - dW/d\rho \text{ is given by} \] 

\[ V_{-}(\rho) = \frac{a(1 + a)}{\rho^{2}} + b \rho^{2} - \frac{2ag}{\rho} + 2ab - 3b \] 

with the condition that \( ag^{2} = -b \). This potential is related to the effective potential \( V(\rho) \) of Eq. (5.13) and the ground state zero-point energy by \( V_{-}(\rho) = 2V(\rho) - 2\epsilon_{g} \). We choose the constants such that the \( \rho \)-dependent terms match and thus find

\[ a = -(\Lambda + 1), \quad b = \beta^{2}, \quad g = \beta/(\Lambda + 1)^{1/2} \] 

and \( \lambda = -ag \). Here the negative value of \( a \) is chosen to ensure the normalizability of the ground-state wave function. The corresponding energy is \( \epsilon_{\rho} = (3 - 2a)b \), and hence we obtain

\[ \epsilon_{\rho} = \frac{\beta}{2} [2(\Lambda + 1) + 3] \] 

with the condition that

\[ \beta = [2Z^{2}(\Lambda + 1)]^{-1}. \] 

Aside from a constant, the ground state wave function is given by

\[ \exp \left[ -\int_{0}^{\rho} W(\rho') d\rho' \right] = \rho^{\lambda} \cdot (1 + g \rho) \exp \left[ -\frac{1}{2} \beta \rho^{2} \right]. \] 

This normalizable, since it vanishes both for \( \rho \to 0 \) and \( \rho \to \infty \).

Figure 1 illustrates the significance of condition (5.18), which restricts values of the force constant \( \beta \) for which the solution holds. This constant vanishes when \( Z \to 0 \) or \( D \to \infty \); in the latter limit, the electrons assume fixed positions.\(^{5}\) For a given finite \( D \) and values of \( Z \) and \( \beta \) not satisfying the condition, the graph of Fig. 1 can be used to obtain upper and lower bounds for the energy. The corresponding range \( \Delta E \) depends on how far the point \((Z, \beta)\) is from the line defined by condition (5.18), although the rectangular construction shown is only valid when the point is close enough to the line.

Excited states can be evaluated by generalizing the ansatz for the superpotential,\(^{22}\) which becomes

\[ W(\rho) = \frac{a}{\rho} + b \rho - \sum_{i=1}^{n} \frac{g_{i}}{1 + g_{i}\rho}, \] 

where the constants \( a, b, \) and \( g_{i} \) are again chosen to make \( \epsilon_{\rho}(\rho) \) match the potential of Eq. (5.13). This yields

\[ \epsilon_{g}(n) = \beta \left( n + \Lambda + \frac{3}{2} \right), \quad \text{with } n = 0, 1, 2, \cdots \] 

with the condition

\[ -(\Lambda + 1)g_{i}^{2} + \beta^{2} \sum_{i=1}^{n} g_{i} = 0 \] 

with \( i = -1, 0, 1, 2, \cdots n \) and \( g_{-1} = 0 \). The parameter \( \lambda \) is given by

\[ \lambda = (\Lambda + 1) \sum_{i=1}^{n} g_{i}. \] 

This Hooke's law atom and variants\(^{23}\) provide examples of the efficacy of combining \( D \) scaling with supersymmetry to obtain results which hold when the parameters satisfy certain relations. Supersymmetry can also be exploited to enhance both the accuracy and simplicity of large-\( D \) expansions.\(^{1}\)

ACKNOWLEDGMENT

We are grateful for support of this work by the Venture Research Unit of BP International Limited.

---

\(^{1}\) E. Witten, Phys. Today 33(7), 38 (1980).

7796
Kais, Herschbach, and Levine: Dimensional scaling as a symmetry operation


