How we think mathematically: A cognitive linguistic approach to understanding mathematical concepts and practices

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Something to ponder

In this mornings two talks it may seem that my talk is about

- Using language to explore cognition

And Michael's talk is about

- Using language to explore communication

But my guess is that they are both about both (to more or less extent) … see if you can tell when the issue is one versus the other … or both at the same time

How do we organize knowledge and create new knowledge

- Metonymy, metaphor and the concept of derivative
  - Zandieh (1997), Zandieh (2000), Zandieh & Knapp (2006)
- Metaphor, functions and linear transformations
  - Zandieh, Ellis, Rasmussen (2012)
- Conceptual blending and proving
  - Zandieh, Roh, Knapp (2012)

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How we organize knowledge

- Women, fire and dangerous things, George Lakoff (1987)
- Metaphors we live by, Lakoff and Johnson (1980)

- *Concept image* is the "set of all mental pictures associated in the students' mind with the concept name, together with all the properties characterizing them" (Vinner & Dreyfus, 1989, p. 356).
- A person's *concept image* for a particular concept is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152).

+ The meanings we have for "the derivative" include both metonymic and metaphoric connections.

	Contexts				
	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
Layers	Slope	Rate	Velocity	Difference Quotient	
Change					
Ratio					
Limit					
Function					

What is metonymy?

- A metonymy occurs when we use "one entity to refer to another that is related to it" (Lakoff & Johnson, 1980, p. 35).

- Lakoff states that a metonymic model consists of two entities A and B that are in the same conceptual structure. B is either part of A or closely related to it.
  - "Compared to A, B is either easier to understand, easier to remember, easier to recognize, or more immediately useful for the given purpose in the given context" (Lakoff, 1987, p. 84).

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If two words or ideas are metonymically related, when can we use the same word for both of them?

- The *strings* played superbly.
- The *White House* called the *Kremlin.*
- The *Bench* heard the evidence.

What helps us understand which meaning is meant?

Strings: strings (e.g. violin strings), stringed instrument (e.g. violin), stringed instrument player (e.g. violinist).

Metonymy condenses concepts

Advantages:

- efficiency or
- interesting emphasis

May cause:

- confusion or
- lack of clarity
- not realizing that two different concepts could be meant

Concept 1: The derivative as a point

Word: The derivative

Concept 2: The derivative as a function

Derivative is the tangent line

[Question] What is the derivative?

Carl: Derivative is the tangent line to the function, isn't it? Isn't it? [pause] It has to do with the tangent and the slope to the graph.

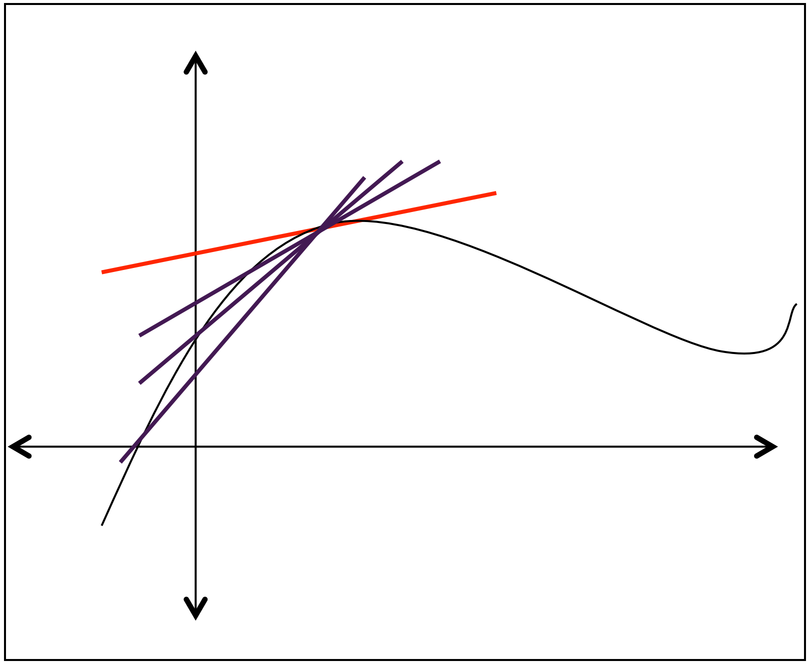
MZ: Make a sentence.

Carl: The derivative is the slope of the tangent line to the graph. Something like that.

[Later] Is the derivative related to a line or linear?

Carl: It's the line, the tangent line. It's the slope of the tangent line is the derivative, so the tangent line to the graph is the derivative as well. They're connected somehow--.

Slope of the tangent line



Classic Derivative Sketch

- Tangent line is explicit
- Slope is implicit

The temperature problem

Let f be a function that for any time x, given in hours, will tell you the outside temperature in degrees Fahrenheit.

a. What do each of the following tell us about the outside temperature?
   - $f'(3) = 4$
   - $f''(3) = -2$
   - $f'(x) = 4$ for $0 \le x \le 3$
   - $f''(x) = -2$ for $3 \le x \le 6$

b. When (during this 6 hours) is the temperature the highest (the lowest)?

c. Approximately what time of day is associated with $x = 0$?

d. How long before the temperature is back to the temperature at time 0?

e. After the student answers the above by their own method, if not done, prompt for the graph of the function on [0,6].

Derivative is the change

- Carl – At any instant in between that interval it's changing 4, but that doesn't make any sense because then you get really small intervals and it becomes a trillion degrees.
- Carl realized that his two statements were contradictory and guessed that his first answer, 4 degrees for the whole interval, was correct.
- Derick – That's implying that at exactly 3 o'clock the temperature increased exactly four degrees Fahrenheit. That's kind of an extreme value don't you think?
- Derick did not recognize that the change is 4 degrees *per hour*.

Using *velocity* to reason about temperature

Derick corrects his misstatement, by relating it to speed.

Derick: It's like the speed of the temperature is 4° in the same way that you take of a car function. At that particular point, that's how fast it's moving. ... So that tells you that it's heating up quite rapidly, but just at that moment.

[...]

Derick: So yeah, it didn't go up 4°, but it's increasing that fast at that particular point. ... If it keeps going up at that constant rate, in an hour it will have gone up 4°. ... It's like the instantaneous speed of the thing.

Metaphor expands concepts and can create new knowledge

Context 1: Slope

Word (Idea):

The derivative

Context 2: Velocity

+ The meanings we have for "the derivative" include both metonymic and metaphoric connections.

	Contexts				
	Graphical	Verbal	Paradigmatic Physical	Symbolic	Other
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A note on terminology: Metaphor versus Metaphorical Expressions

- “Since metaphorical expression in our language are tied to metaphorical concepts… we can use metaphorical linguistic expressions to study the nature of metaphorical concepts and to gain an understanding of the metaphorical nature of our activities” (Lakoff & Johnson, 1980, p. 456).
- Care need be taken before assuming that a metaphorical expression or a cluster of such is an actual conceptual metaphor. (Nuñez, personal communication)

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Concept image of function versus linear transformation

- 10 Students – end of a Linear Algebra course
- In class survey (given to all students in class including 'our' 10 students)
- Interviews – approximately 45-60 minutes
- Typical symbolic example of a linear transformation in this linear algebra class:

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Questions Analyzed
(For this Presentation)

1. In the context of high school algebra, explain in your own words what a function is.

2. In the context of linear algebra, explain in your own words what a transformation is.

3. On a scale from 1-5, to what extent you agree with the following statement: "A linear transformation is a type of function." Explain.

Results

1. Classification of concept image of function and linear transformation
   a. Properties
   b. Computations
   c. Clusters of metaphorical expressions
2. Example of usefulness of this classification

Clusters of Metaphorical Expressions

- Input/ Output
- Traveling
- Morphing
- Mapping
- Machine

Input/ Output

Indicative Phrases: input, output, put in, plug in, get out, take out, accept, receive, returns

- *Jerry: A function f of x = y means that putting x inside would give you a specific output, y.*
- *Gabe: … a function is an equation that accepts an input and returns an output based on that input.*

Traveling

Indicative Phrases: gets sent, goes to, change in the location, reach, go back, get to, moving, direction

- *Adam: A transformation is <u>moving a point or object in a certain direction,</u> like a scalar by a transformation matrix.*

+ Morphing

Indicative Phrases: transform, change, become

- *Donna: Linear transformations to me are more or less something that <u>changes</u> something from one thing to another.*

+ Mapping

Indicative Phrases: assigns, per, for, rule

- *Lawson: [A linear transformation is] a <u>rule that assigns</u> a given input to a certain output or image of the input.*

Machine

Indicative Phrases: acts on, produces, apply, manipulates, operation

- *Nigel: A function is an operation on something.*
- *Gabe: Pretty much anything you toss in here, this is still that transformation should be able to act on it.*

Clusters of Metaphorical Expressions

Cluster	Entity 1	Middle	Entity 2
Input/Output (IO)			
Traveling (Tr)			
Morphing (Mor)			
Mapping (Map)			
Machine (Mach)			

How do they see these as the same?

On a scale from 1-5, to what extent do you agree with the following statement: "A linear transformation is a type of function."

- All 10 students agreed
  - 5 strongly agreed
  - 5 agreed
- But they talk about them differently…

Students' Concept Images

Student	Function	Linear Transformation	How do you see these as the same?
Adam	$P_{equation}$	Tr	IO, Mor, Tr, Comp
Brad	IO, Mor	IO, Mor	IO, Mor
Donna	IO	Mor	Mor, IO
Gabe	$P_{equation}$, IO	$P_{equation}$, Tr	IO, Mach, Mor
Jerry	IO	Mach	IO, Mach
Josh	Comp	Mor, Mach	Comp
Lawson	IO	Map, IO	Map, IO, Mor, Mach
Nigel	Mach	Mor	Mor, Mach
Nila	$P_{equation}$, IO	Mach	$P_{equation}$, P_{VLT}, Mach, Mor
Randall	$P_{equation}$, IO, Map	Comp, Mor	P_{VLT}

Concept image of function

1. In the context of high school algebra, explain in your own words what a function is.

a method that takes an input & spits out an output

Coding: Input/ Output

Concept image of transformation

2. In the context of linear algebra, explain in your own words what a transformation is.

a rule that assigns a given input to a certain output or "image of the input"

Coding: Mapping and Input/ Output

"A linear transformation is a type of function."

Lawson: I agree…Because it essentially does the same thing. So it's like, how I have here a rule that assigns, essentially a function is the same thing, you put in an input, and it manipulates that input and turns it into an output. And that's essentially what a transformation I would say is, because it transforms something into something else.

Coding: Mapping, Input/ Output, Machine, and Morphing

Conceptual blending -- creating new knowledge

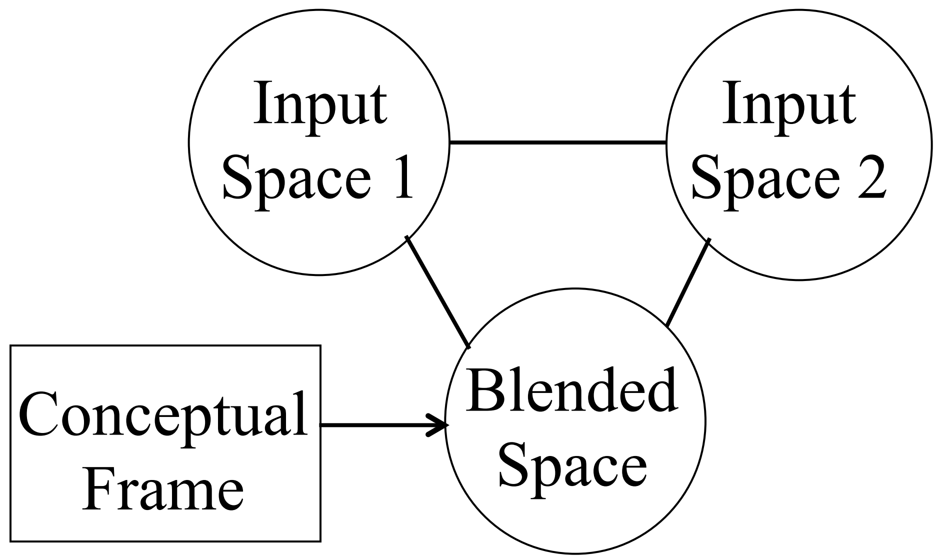
- Fauconnier and Turner (2002)
  - conceptual blending as a powerful unifying theory
  - describes how people think across multiple domains
  - blending "makes possible … diverse human accomplishments … [in] language, art, religion, [and] science [as well as being] indispensable for basic everyday thought" (p. vi).

Conceptual Blending

- Selected references in diverse fields
  - linguistics (Delbecque & Maldonado, 2011)
  - literature (Cook, 2010)
  - philosophy (Fenton, 2008)
  - neural networks (Thagard & Stewart, 2011)
  - mathematics education (Abrahamson, 2009; Gerson & Walter, 2008)
  - mathematics (Núñez, 2005; Lakoff & Nuñez, 2000)
  - physics education (Podolefsky & Finkelstein, 2007; Wittmann, 2010).

+ Conceptual Blending Details

- Mapping
- Completing
- Running



Generic blending diagram

Cow jumps over the moon

- Mapping to the blend
  - Input spaces: (1)animals, (2)moon and sky
- Completing the blend
  - Bringing a jumping frame to the situation
  - May need a nursery rhyme scenario
- Running the blend
  - Imagining the cow taking off from the ground, being over the moon, and landing on the ground on the other side of the moon

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Creating a proof: global insight and detailed structure

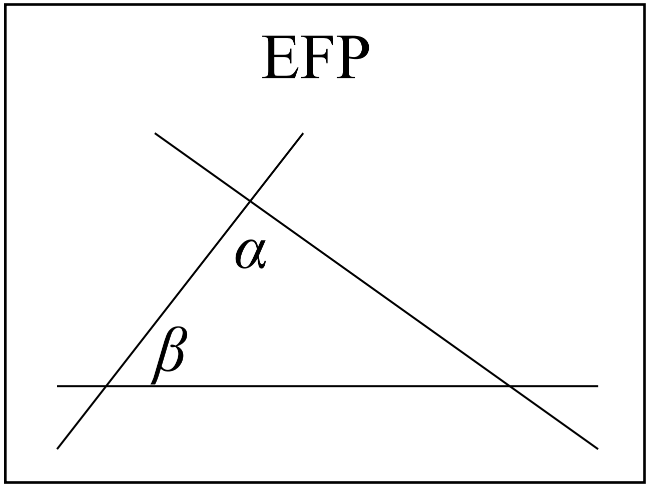
- the *key idea* of the proof (Raman, 2003): the linchpin of the argument that connects informal conviction with the means for producing a formal argument

- For a mathematician, fully understanding a complex mathematical result demands both its intuitive apprehension as a whole and its detailed proof broken down into a series of steps. The former is actually highly valued and commonly viewed as the essence of all creativity. The latter is also valued of course, but often viewed as a way to keep intuition in check and publicly share the results of a discovery. (Fauconnier and Turner, 2000)

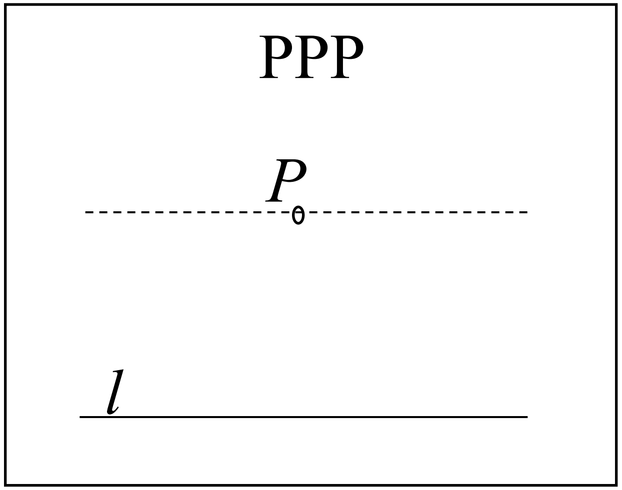
Creating a proof

- Global insight: Where do key ideas come from?
  - Mapping to and running the geometric blend (initial intuitive blend)
- Framing: How do we choose a structure for the proof?
  - Completing the structural blend
- Connections: How do we create the details of the proof?
  - Running the combined blend
- What can go wrong in the compressions involved in blending?

EFP and PPP

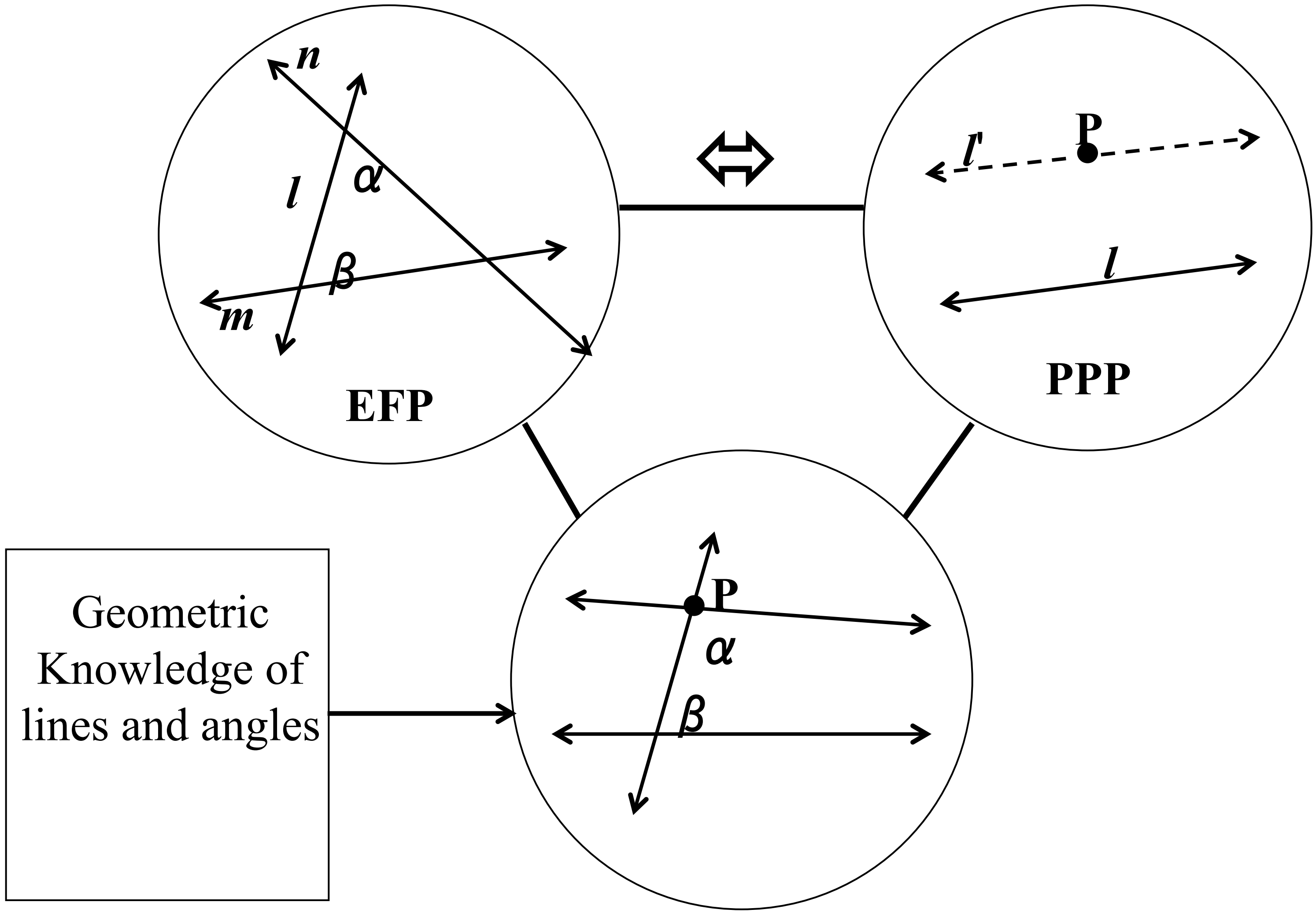


Euclid's Fifth Postulate



Playfair's Parallel Postulate

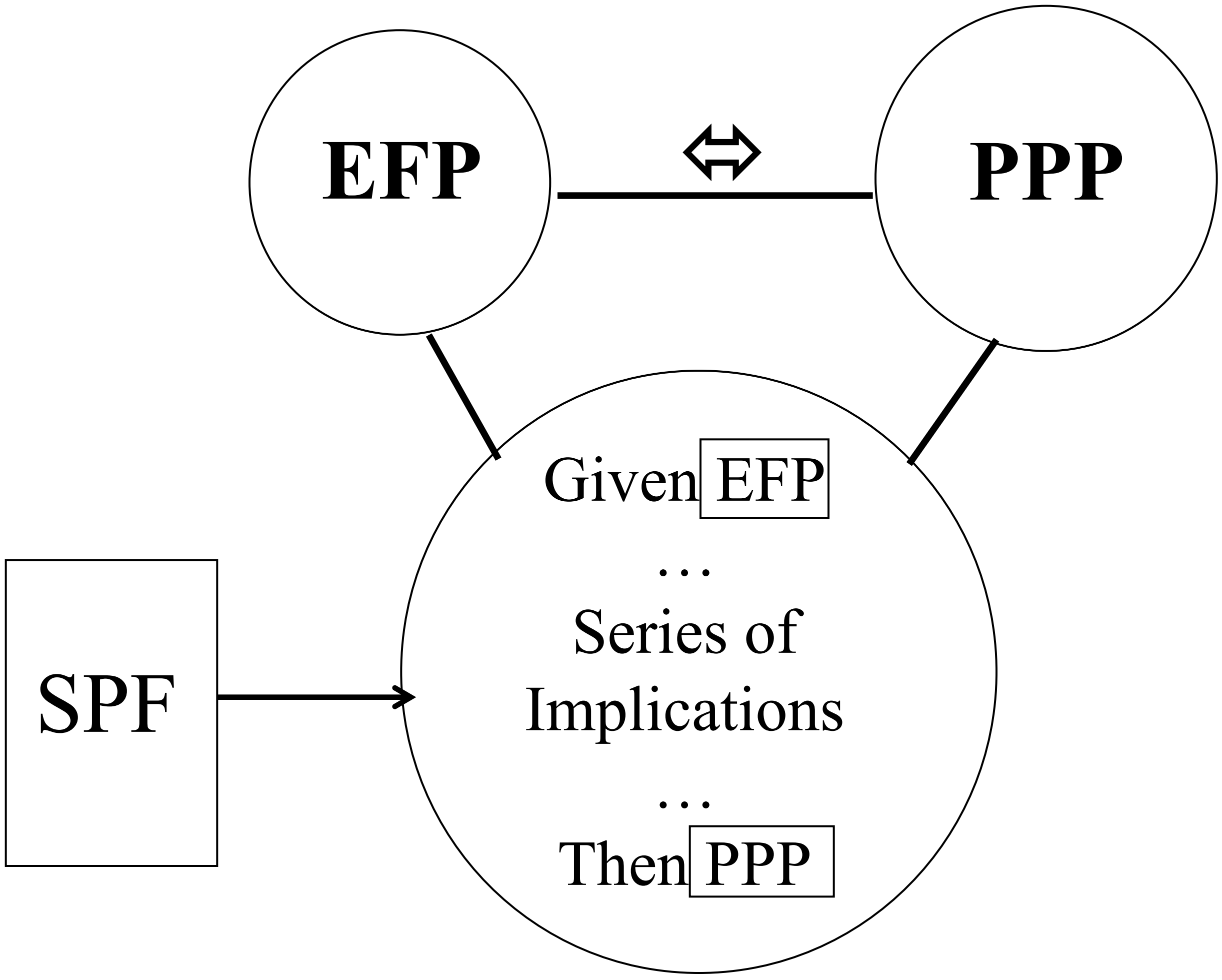
Global insight: Key Geometric Blend (KGB)



Simple Proving Frame (SPF)

Generic SPF	Case 1: $(p \to q)$	Case 2: $(p \to q)$
Given $\boxed{}$ … Series of Implications … Then $\boxed{}$	Given p … Series of Implications … Then q	Given $(p \to q)$ … Series of Implications … Then $(r \to s)$

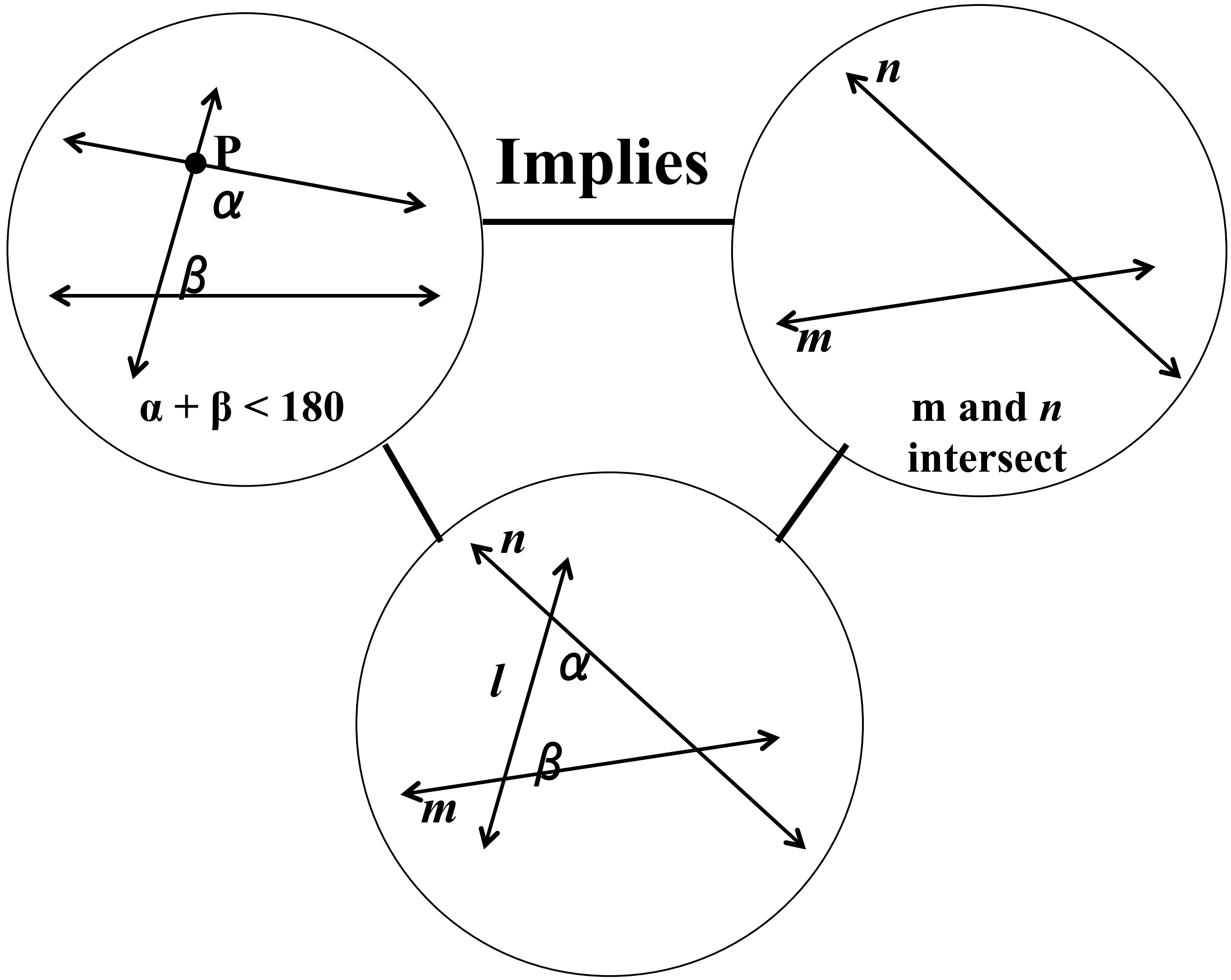
A Structural Blend using SPF



Conditional Implies Conditional Frame (CICF)

Generic CICF	For the case of $(p \rightarrow q) \rightarrow (r \rightarrow s)$	For the case of EFP → PPP
Given … Use … Thus	Given r … Then p Since p and $(p \rightarrow q)$ Then q … Thus s	Given a line and a point … Then $\alpha + \beta < 180$ Since $\alpha + \beta < 180$ *and* EFP Then the lines intersect … Thus there exists a unique line does not intersect.

Blending the Premise and Conclusion



Aspects of Combined Blends

Blends constructed for either direction of the proof: EFP implies PPP (EtoP) or PPP implies EFP (PtoE)

- Geometric blend: Blending the pictures
  - the key geometric blend (KGB) or Stacey's geometric blend (SGB).
- Structural blend: Framing the proof
  - a simple proving frame (SPF) or a conditional implies conditional proving frame (CICF).
- Combined blend: Blends the geometric and structural into a single blend

Combined blends tell the story of the proof creation

Episode		*Main Blend*				*Secondary Blend*			
	Time	**Structure**	**Direction**	**Geometry**	**Presenter**	**Structure**	**Direction**	**Geometry**	**Presenter**
1	9:00	SPF	EtoP	KGB	Paul	CICF	EtoP	KGB	Nate
2	15:13	SPF	EtoP	SGB	Stacey				
3	17:48	SPF	PtoE	KGB	Andrea	CICF	EtoP	KGB	Nate
4	24:38	CICF	EtoP	KGB	Nate	SPF	EtoP	KGB	Paul

Thank you!

- Looking forward to our discussion after coffee …